

# Dependent Defaults and Losses with Factor Copula Models \*

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## Abstract

We introduce a class of flexible and tractable static factor models for the joint term structure of default probabilities, the factor copula models. These high dimensional models remain parsimonious with pair copula constructions, and nest numerous standard models as special cases. With finitely supported random losses, the loss distributions of credit portfolios and derivatives can be exactly and efficiently computed. Numerical examples on collateral debt obligation (CDO), CDO squared, and credit index swaption illustrate the versatility of our framework. An empirical exercise shows that a simple model specification can fit credit index tranche prices.

**Keywords:** credit portfolio, credit derivatives, discrete Fourier transform, factor copula, random loss, survival models.

**JEL Classification:** C10, G12, G13

**AMS Classification (2010):** 60E05, 60E10, 62H05, 62H20, 65T50, 91G20, 91G40, 91G60

## 1 Introduction

We introduce factor copulas to model dependent default times and losses. We directly specify the joint probability of default times, taking as given the marginal default probabilities. Specifically, the default times are assumed to be independent conditional on a latent factor. The joint default probability is given by an explicit expression in terms of conditional copulas. We show that this specification nests all the standard factor models, such as the Gaussian, Archimedean, and stochastic correlation models. In addition, our framework has two main advantages over the existing models. First, the types of dependence between the default times and the latent factor can be highly heterogeneous across entities. Second, new simple and flexible models can be constructed using mixtures and cascades of pair copulas.

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We present a new approach to compute efficiently and exactly the loss distribution of credit portfolios and derivatives on these portfolios. In particular, this allows us to compute the exact payoff distribution of credit portfolio derivatives such as portfolio tranche, collateralized debt obligation (CDO) squared, and credit index swaption. Conditional on the latent factor, the realized individual losses are assumed to be independent from each others and from the default times. This enables us to retrieve the exact portfolio loss distribution using discrete Fourier transform methods. This contrasts with existing approaches that either compute the exact loss distribution using slow recursive methods, or compute an approximate loss distribution by discretizing its support and then applying Fourier techniques. We also suggest the Beta-binomial distribution as a flexible mean to specify the loss given default of each entity.

We explore the versatility of our setup and discuss the impact of different dependence hypothesis on the loss distribution with numerical examples. The discrete Fourier method is shown to be significantly faster than the recursive method, especially when the dimension of the latent factor or of the loss support size is large. We construct a simple model for which the total number of defaults distribution exhibits the features of both highly and little dependent defaults, namely a bump and a fat tail. We compute the loss distribution of a CDO tranche and show that the loss distribution of a portfolio of tranches, also known as CDO squared, may have dramatically different profiles depending on the dependence structure between the underlying tranches. We illustrate the flexibility of the Beta-binomial models for individual loss amounts, and show that the specification of individual losses may also critically affect the portfolio loss distribution.

As an application, we use our approach to fit the market tranche prices on the North America investment grade credit index series 21. We start by exploring various model specifications (from standard copulas to multi-factor models), and introduce a mixture with two Gaussian copulas, parametrized with two correlations and a weight balancing each component. In a static analysis, we find that the mixture outperforms the other models, as it is the only one reproducing the prices of both the junior and senior tranches. Fitting this model for all days in our sample, we further find that the parameters are stable over time. Furthermore, one of the correlations being almost always equal to one, we repeat the exercise by fixing it to 0.999. Interestingly, we find similar results, therefore achieving an almost perfect calibration to all tranches using only two parameters (i.e. the other correlation and the weight).

Although motivated by credit risk applications, we present a generic framework to model dependent defaults and losses in high dimensions that may be useful in other areas of survival analysis.

We now review some of the related literature. Our approach builds on recent advances on the high-dimensional modeling of random variables. When dealing with multivariate data, copulas are attractive, allowing to model separately the marginal distributions and the dependence structure. Unfortunately, few copulas remain practically useful in high-dimensional settings, because common parametric families are often either too flexible, or not enough. An example of the former is the elliptical family, whose members have a number of parameters that grows quadratically with the dimension. Conversely, members of the archimedean family have a small and fixed number of parameters, independently of the dimension. Recently, high-dimensional copulas using a factor structure have been constructed independently by Oh and Patton (2013, 2015) and Krupskii and Joe (2013, 2015). Such approaches alleviate the curse of dimensionality by considering a smaller set of latent variables, conditional upon which the random variables of interest are assumed independent. Arguably the main difference between the methods presented in Oh and Patton (2013, 2015) and Krupskii and Joe (2013, 2015) is that copulas proposed in the former can only be simulated, whereas those in the former admit closed form expressions. In fact, it can be shown the factor copulas from

Krupskii and Joe (2013, 2015) are a special case of pair-copula constructions (PCCs). One of the hot topics of multivariate analysis over the last couple of years, PCCs are flexible representations of the dependence structure underlying a multivariate distribution. Introduced by Bedford and Cooke (2001, 2002) and popularized by Aas et al. (2009), PCCs are decompositions of a joint distribution by considering pairs of conditional random variables. For a given joint distribution, such a construction is not unique, but all possible decompositions can be organized as graphical structures, the so-called PCCs. Assuming the copula linking default times as in Krupskii and Joe (2013, 2015), an interesting aspect of our approach is that it nests the standard models described for instance in Li (2000); Burtshell et al. (2005); Hofert and Scherer (2011) as special cases. Although static by construction, our approach can be incorporated in a dynamic doubly stochastic framework as described in Schönbucher and Schubert (2001) in order to generate stronger default correlation than in pure intensity based models.

To recover the loss distribution, recursive techniques with proportional loss given default have been studied by Andersen et al. (2003); Hull and White (2004), and Fourier approximations are presented in Gregory and Laurent (2003); Laurent and Gregory (2005). The computational performance of the latter approach has been improved for models with a large number of Gaussian factors in Glasserman and Suchintabandit (2012) by using a quadratic approximation technique.

The calibration of tranches on credit portfolios is a daunting task, which is often solved in an ad-hoc way (e.g., by considering a specific model for each tranche). Significant effort have been made to develop consistent models, see Giesecke (2008) for a comparison between top down and bottom up approaches. Standard copula models generally had limited empirical success and non-standard frameworks have been developed, see Hull and White (2006); Kalemianova et al. (2007); Cousin and Laurent (2008); Herbertsson (2008); Fouque et al. (2009); Burtshell et al. (2009); Filipović et al. (2011). In this paper, we develop bottom-up models that are both simple to calibrate and successful at reproducing all the tranche spreads. Furthermore, while the valuation of CDO squared has been considered with simulations in Hull and White (2010); Guillaume et al. (2009), this paper is the first to derive explicitly the loss density of a CDO squared in a factor copula framework.

The realized loss at default on corporate loans and bonds is known to be stochastic, volatile, and negatively correlated with the business cycle. The recovery rates volatility and correlation with default risk is studied, for example, in Altman et al. (2004). These important properties and their impact on the valuation of credit derivatives have been investigated by Andersen and Sidenius (2004); Krekel (2008); Amraoui and Hitier (2008).

The remainder of the paper is structured as follows. Section 2 presents the factor copula framework. Section 3 describes the construction of the individual loss amounts and the computation of the loss distributions. Section 4 contains numerical examples illustrating the performance of our setup and the impact of different dependence hypothesis. The empirical analysis is in Section 5. Section 6 concludes. The appendix contains the proofs, additional results on standard factor copula models, and the pricing formulas for tranches, credit index swaps, and credit index swaption.

## 2 The factor copula framework

We consider  $N$  entities. For each  $j = 1, \dots, N$  let  $p_{j,t}$  be a non-decreasing deterministic function satisfying  $p_{j,0} = 0$  and  $\lim_{t \rightarrow \infty} p_{j,t} = 1$  for all  $0 < t < \infty$ . We define the default time  $\tau_j$  of entity  $j$  as follow

$$\tau_j := \inf\{t \geq 0 : U_j \leq p_{j,t}\},$$

where  $U_j$  is a uniform random variable on the unit interval. Hence, the function  $p_{j,t}$  is equivalent to the marginal default probability of entity  $j$

$$\mathbb{P}[\tau_j \leq t] = \mathbb{P}[U_j \leq p_{j,t}] = p_{j,t}.$$

When  $p_{j,t}$  is absolutely continuous with respect to time, it has the following representation

$$p_{j,t} = 1 - e^{-\int_0^t \lambda_{j,s} ds} \quad (1)$$

for some non-negative default intensity function  $\lambda_{j,s}$ .

Note that, in this setup, the random vector  $U = (U_1, \dots, U_N)$  is the only stochastic object. We recall that its probability distribution is by construction a copula.

**Definition 2.1.** *A copula  $C_U$  is the probability distribution of a random vector  $U$  taking values on the hypercube  $[0, 1]^N$  and having uniform marginal distributions.*

In other words, if for any vector  $(u_1, \dots, u_N) \in [0, 1]^N$  the random vector  $U \in [0, 1]^N$  is such that  $\mathbb{P}[U_j \leq u_j] = u_j$  for each  $j$ , then its joint distribution is called a copula and we write

$$C_U(u_1, \dots, u_N) = \mathbb{P}[U_1 \leq u_1, \dots, U_N \leq u_N]. \quad (2)$$

The following lemma shows that for  $(t_1, \dots, t_N) \in \mathbb{R}_+^N$ , there exists a simple expression linking joint to marginal default probabilities using the copula  $C_U$  of  $U$ .

**Lemma 2.2.** *The joint default probability is given by*

$$\mathbb{P}[\tau_1 \leq t_1, \dots, \tau_N \leq t_N] = C_U(p_{1,t_1}, \dots, p_{N,t_N}). \quad (3)$$

A direct construction of high-dimensional copulas amounts at trading-off model complexity and tractability. This is somewhat problematic, because the usual parametric families contain either too many (e.g., in the case of implicit copulas extracted from known multivariate distributions), or too few (e.g., in the case of Archimedean copulas built using a continuous and nonincreasing  $N$ -monotone generator) parameters. Furthermore, as we will show in Section 3 when pricing complex financial derivatives, the notion of conditional independence (on a set of latent factors) allows us to obtain a flexible yet tractable class of models. Hereinafter we therefore focus on the so-called factor copulas.

## 2.1 One-factor copulas

A one-factor copula model is constructed by assuming that there exists a latent factor  $V$  such that, conditional on the realization of  $V$ , the coordinates of the random vector  $U$  are independent. Further assuming that  $V$  is uniformly distributed on the unit interval<sup>1</sup>, it means that

$$\mathbb{P}[U_1 \leq u_1, \dots, U_N \leq u_N \mid V = v] = \prod_{j=1}^N \mathbb{P}[U_j \leq u_j \mid V = v] \quad (4)$$

for any vector  $(u_1, \dots, u_N) \in [0, 1]^N$  and for any  $v \in [0, 1]$ . The following proposition shows that such an assumption yields a simple decomposition in terms of bivariate copulas for  $C_U$ . We refer to such copula as one-factor copula.

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<sup>1</sup>This is without loss of generality as the latent factor could be mapped to such a  $V$  using the probability integral transform if it was not the case.

**Proposition 2.3** (One-factor copula). *For  $j = 1, \dots, N$ , let  $C_{U_j, V}$  denote the joint distribution of  $U_j$  and  $V$ , that is  $\mathbb{P}[U_j \leq u_j, V \leq v] = C_{U_j, V}(u_j, v)$ . If the coordinates of  $U$  are independent conditionally on  $V$ , then*

$$C_U(u_1, \dots, u_N) = \int_{[0,1]} \prod_{j=1}^N C_{U_j|V}(u_j | v) dv, \quad (5)$$

where, for all  $j = 1, \dots, N$ ,

$$C_{U_j|V}(u_j | v) = \frac{\partial C_{U_j, V}(u_j, v)}{\partial v}$$

are the so-called *h-functions*.

The *h-functions* have been introduced by Aas et al. (2009) while studying the pair-copula decomposition of a general multivariate distribution: if  $C_{U_j, V}(u_j, v) = \mathbb{P}[U_j \leq u_j, V \leq v]$ , then  $C_{U_j|V}(u_j | v) = \mathbb{P}[U_j \leq u_j | V = v]$ .

Note that  $C_{U_j, V}(u, v) = uv$  implies  $C_U(u_1, \dots, u_N) = \prod_{j=1}^N u_j$ . In other words, if  $U_j$  is independent from  $V$ , then it is also independent from  $U_k$  for all  $k \in \{1, \dots, j-1, j+1, \dots, N\}$ , which means that the coordinates of  $U$  depend on each other only through the factor  $V$ .

**Example 2.4.** *The Gaussian model of Li (2000) is a one-factor copula obtained by using for all  $j$*

$$C_{U_j, V}(u_j, v; \rho) = \Phi_2(\Phi^{-1}(u_j), \Phi^{-1}(v); \rho),$$

which implies

$$C_{U_j|V}(u_j | v; \rho) = \Phi\left(\frac{\Phi^{-1}(u_j) - \rho\Phi^{-1}(v)}{1 - \rho^2}\right),$$

and

$$C_U(u_1, \dots, u_N; \rho) = \int_0^1 \prod_{j=1}^N \Phi\left(\frac{\Phi^{-1}(u_j) - \rho\Phi^{-1}(v)}{1 - \rho^2}\right) dv,$$

where  $\Phi(\cdot)$  is the standard normal distribution and  $\Phi_2(\cdot, \cdot; \rho)$  is the bivariate normal distribution with correlation  $\rho$ .

Observe that the specification in Proposition 2.3 is far more flexible, since one could build a model using, for each entity, a different bivariate copula, for which countless well-studied parametric families exist (see Schepsmeier and Stöber (2014)).

Beyond such parametric families, a simple way to increase the modeling flexibility while preserving analytical tractability is to combine different bivariate copulas. Through the following definition, mixture distributions enrich considerably the one-factor copulas.

**Definition 2.5.** *Let  $K$  be a positive integer,  $C_{U_j, V}$  is a mixed bivariate copula if there exists  $K$  copulas  $C_{U_j, V}^k$ ,  $K$  positive weights  $w_k > 0$  such that  $\sum_{k=1}^K w_k = 1$ , and*

$$C_{U_j, V}(u_j, v) = \sum_{k=1}^K w_k C_{U_j, V}^k(u_j, v). \quad (6)$$

One way to interpret this expression is Bayesian, namely assuming that the dependence between the random variable  $U_j$  and the factor  $V$  is uncertain and follows the distribution  $C_{U_j, V}^k$  with probability  $w_k$ . The corresponding  $h$ -function still has a simple expression, as we have

$$C_{U_j|V}(u_j | v) = \sum_{k=1}^K w_k C_{U_j|V}^k(u_j | v).$$

Of particular interest for risk management applications, the joint distribution of default times conditional on a subset of realized default times is obtained as a simple modification of Equation (5). Let  $\mathcal{I} = \{1, \dots, N\}$  and  $\mathcal{D} \subset \mathcal{I}$  denote respectively the entire set and a subset of entities. The following proposition shows that the joint default distribution conditional on the defaults of all the entities in  $\mathcal{D}$  also has a simple representation.

**Proposition 2.6.** *In a one-factor copula model, the joint default distribution conditional on  $\tau_k = t_k$  for  $k \in \mathcal{D}$  is*

$$\mathbb{P}[\tau_1 \leq t_1, \dots, \tau_N \leq t_N | \tau_k = t_k : k \in \mathcal{D}] = \int_{[0,1]} \prod_{j \in \mathcal{I} \setminus \mathcal{D}} C_{U_j|V}(p_{j,t_j} | v) \prod_{k \in \mathcal{D}} c_{U_k,V}(p_{k,t_k}, v) dv \quad (7)$$

where

$$c_{U_j,V}(u, v) = \frac{\partial^2 C_{U_j,V}(u, v)}{\partial u \partial v}$$

is the density of the bivariate copula  $C_{U_j,V}$ .

Although the default times are correlated, conditioning on a subset of defaulted entities does not significantly complexify the expression for the joint distribution of the surviving entities. This result may be of particular interest to compute the loss distribution of a credit portfolio conditional on the default time of a specific entity which in turn could be used to compute the Credit Valuation Adjustment with respect to this entity.

## 2.2 Multi-factor copulas

In this section we consider a  $d$ -dimensional random vector of latent factors  $V = (V_1, \dots, V_d)$ . We assume that  $V$  takes values on the hypercube  $[0, 1]^d$  and has uniform marginal distributions. The joint distribution of  $V$  is by definition a copula that we denote  $C_V$ . The following proposition shows that the one-factor framework extends to a multi-factor one.

**Proposition 2.7** (Multi-factor copula). *For  $j = 1, \dots, N$ , let  $C_{U_j,V}$  denote the joint distribution of  $U_j$  and  $V$ , that is  $\mathbb{P}[U_j \leq u_j, V \leq v] = C_{U_j,V}(u_j, v)$ . If the coordinates of  $U$  are independent conditionally on  $V$ , then*

$$C_U(u) = \int_{[0,1]^d} \prod_{j=1}^N C_{U_j|V}(u_j | v) dC_V(v) \quad (8)$$

where, for all  $j = 1, \dots, N$ ,

$$C_{U_j|V}(u_j | v) = \frac{\partial^d C_{U_j,V}(u_j, v)}{\partial v_1 \dots \partial v_d}.$$

Although (8) appears to be similar to (5), it is arguably more complicated. The reason is that, instead of being bivariate, each  $C_{U_j, V}$  has dimension  $d + 1$ . However, the multi-factor framework simplifies under the assumption of independent latent factors  $V$  as shown in the following proposition. We denote the function composition with the symbol  $\circ$ , that is  $f(g(x)) = f \circ g(x)$  for any real valued functions  $f$  and  $g$ .

**Corollary 2.8** (Copulas with independent factors). *If  $C_V(v) = \prod_{j=1}^d v_j$ , then*

$$C_U(u_1, \dots, u_n) = \int_{[0,1]^d} \prod_{j=1}^N C_{U_j|V_1}(\cdot|v_1) \circ \dots \circ C_{U_j|V_d}(u_j|v_d) dv, \quad (9)$$

where  $C_{U_j, V_k}$  is a bivariate copula for  $j \in 1, \dots, d$  and  $k = 1, \dots, d$ .

As observed in Krupskii and Joe (2013), the recursive decomposition from (9) is a particular case of pair-copula constructions (PCCs), which are representations of flexible joint distributions as cascade products of bivariate copulas and marginals. For more details on the subject, we refer to Bedford and Cooke (2001, 2002), which proposed a graphical model to help organizing PCCs, or Aas et al. (2009), which popularized them by developing efficient computational algorithms for their inference and simulation.

This construction is interesting for several reasons. First, it is a parsimonious way to model a complex multivariate dependencies. Second, the hierarchical structure, which can be represented as a graphical model, has a visual interpretation (see Figure 1). Third, because the integrand in (9) is a simple recursion, it can be vectorized in a computationally efficient manner.

Finally, it should be noted that the number of latent factors is also the dimension of the hypercube on which the product of conditional copulas has to be integrated to retrieve the joint default probability. One should therefore balance between higher modeling flexibility and lower computational cost.

### 2.3 Comparison with standard factor models

In this section, we show that standard static models can be rewritten explicitly as factor copula models. One usually considers a random vector  $Y = (Y_1, \dots, Y_N) \in \mathbb{R}^N$  along with a deterministic and componentwise non-decreasing vector  $y_t = (y_{1,t}, \dots, y_{N,t}) \in \mathbb{R}^N$ . For instance,  $Y$  can represent the values of  $N$  firms and  $y_t$  the corresponding default barriers<sup>2</sup>. The default time  $\tau_j$  of firm  $j$  is then defined as the first time its value is below its default barrier, that is

$$\tau_j = \inf\{t \geq 0 : Y_j \leq y_{jt}\}.$$

Additionally, standard factor models are constructed by decomposing the stochastic behavior of the firm value into a systemic and an idiosyncratic component. In other words, one assumes the existence of a random vector  $X \in \mathbb{R}^d$  and  $N$  variables  $\epsilon_j$  for  $j \in \{1, \dots, N\}$ , such that  $Y_j$  is a function  $X$  and  $\epsilon_j$ , that is

$$Y_j = f_j(X, \epsilon_j)$$

for some  $(d + 1)$ -dimensional function  $f_j$  taking values on  $\mathbb{R}_+$ .

Let  $F_{Y_j}$ ,  $F_X$ , respectively  $F_{Y_j}^{-1}$ ,  $F_X^{-1}$ , denote the distributions of  $Y$  and  $X$ , respectively their inverse, and  $F_{Y_j|X}$  denote the conditional distribution of  $Y_j$  given  $X$ . The following proposition shows that any standard factor model is equivalent to a specific factor copula model.

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<sup>2</sup>While firm values and default barriers are usually positive, this can be resolved by using a monotonic transformation of  $Y$  and  $y_t$  without affecting the results that follow. For instance, with  $\tilde{Y} = e^Y$  and  $\tilde{y}_t = e^{y_t}$ , it is clear that  $\mathbb{P}[Y_j \leq y_{j,t}] = \mathbb{P}[\tilde{Y}_j \leq \tilde{y}_{j,t}]$  and that the copulas of  $Y$  and  $\tilde{Y}$  are the same

**Theorem 2.9.** *A standard factor model is a factor copula model with marginal default probabilities  $p_{j,t} = F_{Y_i}(y_{j,t})$  and conditional copulas*

$$C_{U_j|V}(u | v) = F_{Y_j|X}(F_{Y_j}^{-1}(u) | (F_{X_1}^{-1}(v_1), \dots, F_{X_N}^{-1}(v_N))),$$

for  $j = 1, \dots, N$ , and where the copula of  $V$  is given by

$$C_V(v) = F_X(F_{X_1}^{-1}(v_1), \dots, F_{X_N}^{-1}(v_N)).$$

Furthermore, if the functions  $F_X$  and  $F_{Y_j}$  for all  $j = 1, \dots, N$  are continuous, then the copulas  $C_V$  and  $C_{U_j|V}$  for all  $j = 1, \dots, N$  are unique.

**Example 2.10.** *The Gaussian model described in Example 2.4 is obtained by writing, for  $j \in \{1, \dots, N\}$ ,*

$$Y_j = \rho X + \sqrt{1 - \rho^2} Z_j,$$

and

$$y_{j,t} = \Phi(p_{j,t}),$$

where  $X, Z_1, \dots, Z_N$  are i.i.d.  $N(0, 1)$  random variables.

In Appendix B, we derive the factor copula representation of other popular models such as the Stochastic correlation, the  $t$ -Student, the Archimedean models, and the Gaussian-Mixture. However, while  $C_{U_j|V}$  and  $C_V$  sometimes admit such closed-form expressions, it is clear that the marginal distributions are irrelevant. Instead, working directly with copulas offers more modeling flexibility while ensuring tractability.

Having described the construction of the joint distribution of default times, we now turn our attention toward the second element of our framework: the modeling of the losses given default. In the next section, we introduce a class of discrete loss distributions which can be computed in quasi-closed form.

### 3 Discrete loss distributions

We define the time- $t$  loss  $L_t$  on a portfolio composed of securities written on  $N$  different obligors as

$$L_t = \sum_{j=1}^N \ell_j \mathbb{1}_{\{\tau_j \leq t\}} = \sum_{j=1}^N \ell_j \mathbb{1}_{\{U_j \leq p_{jt}\}}, \quad (10)$$

where  $\ell_j$  is the possibly random loss amount experienced when obligor  $j$  defaults, and  $\mathbb{1}_{\{\tau_j \leq t\}}$  is the default indicator of obligor  $j$ . In this section, we make two assumptions on  $\ell_j$  to preserve the tractability of the portfolio loss distribution, and to enable efficient numerical techniques.

First, we assume that  $\ell_j$  is  $V$ -conditionally independent of both  $\ell_k$  for  $k \neq j$  and  $U$  (or equivalently  $\tau$ ), that is

$$\mathbb{P}[U \leq u, \ell \leq x | V = v] = \prod_{j=1}^N C_{U_j|V}(u_j | V = v) \mathbb{P}[\ell_j \leq l_j | V = v],$$

with  $\ell = (\ell_1, \dots, \ell_N)$ , and for any  $u \in [0, 1]^N$ ,  $v \in [0, 1]^d$  and  $l \in \mathbb{R}_+^N$ . As in the case of the joint distribution of default times, the  $V$ -conditional probabilities can be arbitrarily specified. Hence,



the conditional independence property does not preclude some dependence between default rates and loss given default.

Second, as in Andersen et al. (2003); Andersen and Sidenius (2004); Hull and White (2004), we assume that the losses are discrete. More specifically, we let  $\delta \in \mathbb{R}_+$  be the common loss unit, such that each  $\ell_j$  has a discrete support starting at zero and with mesh  $\delta$ , that is

$$\ell_j \in \{0, \delta, 2\delta, \dots, m_j\delta\}, \quad j = 1, \dots, N$$

for some integer  $m_j \in \mathbb{N}$ . Hence, the portfolio loss distribution also has a discrete support with the same mesh  $\delta$ , that is

$$L_t \in \{0, \delta, 2\delta, \dots, M\delta\}$$

where  $M = \sum_{i=1}^N m_i$ . Although  $\delta$  is an arbitrary constant, it can be as fine as required in order to mimick the discreteness of real-world prices. For instance, assuming that the granularity of prices is in cents (i.e.,  $\delta = 0.01\$$ ) and that the notional of each contract is 1\$, then  $m_j = 100$  and  $M = N \times 100$ .

In the next section, we describe our method to compute the distribution of  $L_t$  in quasi-closed form using discrete Fourier inversion.

### 3.1 Portfolio loss distribution

In this section, we show that the portfolio loss distribution has an almost closed-form expression that can be efficiently computed numerically. Recall that, for a discrete and finitely supported random variable  $X \in \{0, 1, \dots, M\}$  admitting a characteristic function  $\phi_X(u) = \mathbb{E}[e^{iuX}]$ , its distribution can be represented as a finite sum

$$\mathbb{P}[X = k] = \frac{1}{M+1} \sum_{m=0}^M \phi_X\left(\frac{2\pi m}{M+1}\right) e^{-\frac{2\pi i k m}{M+1}}$$

Therefore, if the characteristic function of the loss distribution admits a closed-form expression, so does the loss distribution itself. Using the  $V$ -conditional independence, the following proposition shows that the characteristic function of the loss admits a simple expression. To improve the clarity of the formulas, we work with the normalized losses

$$\ell_j \delta^{-1} \in \{0, 1, \dots, m_j\}$$

and normalized portfolio loss

$$L_t \delta^{-1} \in \{0, 1, \dots, M\}.$$

**Proposition 3.1.** *The characteristic function of the normalized portfolio loss  $L_t \delta^{-1}$  is given by*

$$\phi_{L_t}(u) = \mathbb{E}\left[e^{iuL_t \delta^{-1}}\right] = \int_{[0,1]^d} \prod_{j=1}^N (1 - p_{j,t}(v) + p_{j,t}(v)\phi_{\ell_j}(u, v)) dC_V(v),$$

for any time  $t \geq 0$  and for  $u \in \mathbb{R}$ , where  $p_{j,t}(v) = C_{U_j|V}(p_{j,t} | v)$  is the conditional default probability of  $j$ ,  $p_{j,t}$  is the unconditional default probability of  $j$  defined by Equation 1, and

$$\phi_{\ell_j}(u, v) = \sum_{k=0}^{n_j} \mathbb{P}[\ell_j = \delta k | V = v] e^{iuk}$$

the  $V$ -conditional characteristic function of  $\ell_j \delta^{-1}$ .

The characteristic function is therefore explicit, up to the integral over the compact set  $[0, 1]^d$  which can be efficiently computed for reasonably large  $d$  using, for example, Legendre quadrature. The following lemma is a reminder that, since the support of the portfolio loss distribution is discrete and finite, we can compute it without approximation as the discrete Fourier transform of its the characteristic function.

**Lemma 3.2.** *The probability distribution of the portfolio loss is given by*

$$\mathbb{P}[L_t = k\delta] = \frac{1}{M+1} \sum_{m=0}^M \phi_{L_t}(\mu m) e^{-i\mu k m} \text{ for } k \in \{0, \dots, M\}, \quad (11)$$

with  $\mu = 2\pi/(M+1)$  and  $\phi_{L_t}(\cdot)$  is the characteristic function of  $L_t\delta^{-1}$ .

Note that calculating directly this distribution is a combinatorial problem whose complexity is increasing exponentially fast with  $M$ . Equipped with Lemma 3.2, the computation boils down to an application of the Fast Fourier Transform (FFT) algorithm, which is of significant practical importance as long as evaluating the characteristic function is efficient.

As mentioned above, the assumption of loss unit and discretely supported portfolio losses appears already in Andersen et al. (2003); Andersen and Sidenius (2004); Hull and White (2004), where the distribution is computed without approximation by a recursive algorithm. However, as will be shown in Section 4.1, the computational cost of this recursion increases much faster with both the support size and the number of factors than that of our approach.

The discrete Fourier inversion in Lemma 3.2 differs from the continuous Fourier inversion described in Laurent and Gregory (2005); Burtschell et al. (2009) which aims to approximate a continuous loss distribution. Since our approach provides quasi-closed expressions for the loss distribution, its scope is much wider, allowing notably the pricing of CDO squared and Credit Index Options without simulations.

**Remark 3.3.** *In the doubly stochastic framework of Schönbucher and Schubert (2001), the marginal default intensities are also driven by stochastic processes. In this case, the characteristic function is usually intractable, making the approach significantly more involved numerically. However, recent advances on polynomial models offer new possibilities. For instance, consider the linear credit risk model from Akerer and Filipović (2016) for the individual survival processes along with polynomial pair copulas (e.g., Bernstein copulas or polynomial approximations of bivariate copulas, see Sancetta and Satchell (2004)). Thanks to the analytical tractability of the polynomial diffusions' conditional higher moments (see Filipović and Larsson (2016)), the computational efficiency is then preserved.*

In the next section, we show that our framework allows us to price in quasi-closed form products as complex as tranches on credit portfolios, portfolios of such tranches, or credit index swaptions. While the market for some is booming (e.g., credit index swaption), other may have fallen out of fashion (e.g., CDO squared). Therefore, we emphasize that such examples are meant to illustrate to potential of combining factor copulas with discretely supported losses given default.

## 3.2 Pricing multi-name credit derivatives

### 3.2.1 Tranches and CDO squared

In this section, we show that, with our framework, the loss distribution of more complex portfolios can also be retrieved explicitly for any horizon of time. We start with reminders on credit index tranches and CDO squared, and derive their loss distributions.

A tranche on a credit portfolio is a derivative that pays a fraction of the realized portfolio losses above the attachment point  $a$  and below the detachment point  $b$  with  $0 \leq a < b$ , in exchange of regular payments functions of the effective tranche width. Define the tranche loss as

$$\mathcal{T}_t^{a,b} := \min \{ \max \{ L_t - a, 0 \}, b - a \}, \quad (12)$$

and denote  $\epsilon_a := \delta - (a \bmod \delta)$ . The following shows that the knowledge of the probability distribution of  $L_t$  implies that of  $\mathcal{T}_t^{a,b}$ .

**Proposition 3.4.** *The tranche loss  $\mathcal{T}_t^{a,b}$  has a discrete support and its probability mass function is given by*

$$\begin{aligned} \mathbb{P} \left[ \mathcal{T}_t^{a,b} = 0 \right] &= \sum_{m=0}^{\lfloor a/\delta \rfloor} \mathbb{P} [L_t = m\delta], \\ \mathbb{P} \left[ \mathcal{T}_t^{a,b} = b - a \right] &= \sum_{m=\lceil b/\delta \rceil}^M \mathbb{P} [L_t = m\delta], \\ \text{and } \mathbb{P} \left[ \mathcal{T}_t^{a,b} = \epsilon_a + k\delta \right] &= \mathbb{P} [L_t = (k + \lceil a/\delta \rceil)\delta], \end{aligned}$$

for any  $k \in \mathbb{N}$  such that  $0 < \epsilon_a + k\delta < b - a$ , and where  $\lfloor x \rfloor$  (respectively  $\lceil x \rceil$ ) denotes the closest integer smaller (respectively larger) than  $x$ .

Similarly, a portfolio composed of multiple tranches from (potentially different) portfolios is known as a CDO squared. As for the tranche, its loss distribution can be computed explicitly, even when the defaults of obligors composing the different portfolios are assumed to be dependent. More formally, let us consider  $K$  tranches on portfolios written on (potentially different) obligors. For  $k \in \{1 \dots, K\}$ , we denote by  $\mathcal{T}_{k,t}^{a_k,b_k}$  and  $L_{k,t}$  the  $k$ -th tranche and portfolio loss, with  $a_k$  and  $b_k$  the  $k$ -th tranche attachment and detachment points. The CDO-squared loss (or simply squared loss) is

$$\mathcal{L}_t = \sum_{k=1}^K \mathcal{T}_{k,t}^{a_k,b_k}.$$

Assume that for all  $k$ , we have

$$a_k \bmod \delta = 0 \quad \text{and} \quad b_k \bmod \delta = 0. \quad (13)$$

Then, each of the tranche losses as well as the squared loss have a discrete state space

$$\begin{aligned} \mathcal{T}_{k,t}^{a_k,b_k} &\in \{0, \delta, 2\delta, \dots, b_k - a_k\} \text{ for } k \in \{1 \dots, K\}, \\ \text{and } \mathcal{L}_t &\in \{0, \delta, 2\delta, \dots, M_K\delta\}, \end{aligned}$$

where  $M_K = \sum_{k=1}^K (b_k - a_k)/\delta$ .

**Corollary 3.5.** *If Equation (13) holds for  $k = 1, \dots, K$ , then the characteristic function of the squared loss is*

$$\phi_{\mathcal{L}_t}(u) = \int_{\mathbb{R}^d} \prod_{k=1}^K \phi_{\mathcal{T}_{k,t}}(u, v) dC_V(v)$$

where

$$\phi_{\mathcal{T}_{kt}}(u, v) = \sum_{n=1}^{(b_k - a_k)/\delta} \mathbb{P} \left[ \mathcal{T}_{kt}^{a_k, b_k} = n\delta \mid V = v \right] e^{iun}$$

is the  $V$ -conditional characteristic function of  $\mathcal{T}_{kt}\delta^{-1}$ .

To compute  $\phi_{\mathcal{T}_{kt}}$  one may use Proposition 3.4 applied to the  $V$ -conditional portfolio loss distribution, namely  $\mathbb{P}[L_{kt} = m\delta \mid V = v]$ . Applying Lemma 3.2 with  $\mathcal{L}_t$  replacing  $L_t$ , one finally obtains the distribution of the squared loss. With the distribution of the squared loss, one can then price derivatives such as tranches on a portfolio of tranches.

### 3.2.2 Credit index swaption

A credit index swaption is an option on a credit index swap. Whereas tranches could be priced using the portfolio loss distribution only, the optionality embedded in such an option necessitates the joint distribution of the total number of defaulted entities and of the total loss, see Appendix C for details on the pricing formulas. Letting  $N_t$  be the number of defaulted entities at time  $t$ , that is

$$N_t = \sum_{j=1}^N \mathbb{1}_{\{\tau_j \leq t\}}, \quad (14)$$

we need  $\mathbb{P}[N_t = n, L_t = \delta k]$  for all  $n = 0, \dots, N$ ,  $k = 0, \dots, M$ , and  $t > 0$ . The following lemma provides a generic expression for the joint distribution of  $(N_t, L_t)$ .

**Proposition 3.6.** *The joint distribution of  $(N_t, L_t)$  is given by*

$$\mathbb{P}[N_t = n, L_t = \delta k] = \sum_{j=0}^N \sum_{l=0}^M \frac{\phi_{N_t, L_t}(\mu j, \nu l) e^{-i\mu n j} e^{-i\nu k l}}{(1+N)(1+M)}$$

with  $\mu = 2\pi/(M+1)$ ,  $\nu = 2\pi/(N+1)$ , and

$$\phi_{N_t, L_t}(x, y) = \int_{[0,1]^d} \prod_{j=1}^N (1 - p_{j,t}(v) + p_{j,t}(v) \phi(x, y, v)) dC_V(v)$$

where  $p_{j,t}(v)$  is as in Proposition 3.1, and

$$\phi(x, y, v) = \sum_{k=0}^{n_j} \mathbb{P}[\ell_j = \delta k \mid V = v] e^{i(x+yk)}.$$

Note that this results requires a two-dimensional discrete Fourier transform inversion as described in the proof. One may observe that  $\phi(x, y, v)$  is the  $V$ -conditional characteristic function of  $x + y\ell_j\delta^{-1}$  evaluated at one, that is

$$\phi(x, y, v) = \mathbb{E} \left[ e^{i(x+y\ell_j\delta^{-1})} \mid V = v \right],$$

and that  $\phi_{N_t, L_t}(x, y)$  is the characteristic function of  $(N_t, L_t\delta^{-1})$  evaluated at  $(x, y)$ , that is

$$\phi_{N_t, L_t}(x, y) = \mathbb{E} \left[ e^{i(xN_t + yL_t\delta^{-1})} \right].$$

**Remark 3.7.** When the loss amounts  $\ell_j$  are homogeneous and independent from  $V$ , then the following more direct calculation can be applied

$$\mathbb{P}[N_t = n, L_t = \delta k] = \mathbb{P}[L_t = k\delta \mid N_t = n] \mathbb{P}[N_t = n]$$

where  $\mathbb{P}[N_t = n]$  can be computed as in Lemma 3.2, and where

$$\mathbb{P}[L_t = k\delta \mid N_t = n] = \mathbb{P}\left[\sum_{j=1}^n \ell_j = k\delta\right]$$

may also be derived using the discrete Fourier transform.

Note that, up to this point, we left unspecified the  $V$ -conditional distribution of the loss amounts. In the next section, we suggest a flexible specification for  $\mathbb{P}[\ell_j = \delta k \mid V = v]$  for  $k \in \{0, \dots, m_j\}$  and  $j \in \{1, \dots, N\}$ , which is required to compute the characteristic function of the portfolio loss in Proposition 3.1 .

### 3.3 Beta-binomial loss amounts

In this section, we assume that the loss amount distribution of each obligor can be dependent on the default times and others loss amounts. For each  $j = 1, \dots, N$ , we let the loss amount  $\ell_j$  take value in a set of the form

$$\ell_j \in \{b_j\delta, (a_j + b_j)\delta, \dots, (n_j a_j + b_j)\delta\} \subset \{0, \delta, 2\delta, \dots, m_j\delta\}$$

with the integers  $a_j, b_j, n_j \in \mathbb{N}$  such that  $n_j a_j + b_j = m_j > 0$ . Note that the two sets are equivalent when  $a_j = 1$  and  $b_j = 0$ . The Beta-binomial model is obtained by assuming that the  $V$ -conditional distribution of the loss amount increment  $(\ell_j \delta^{-1} - b_j)/a_j$  is a Beta-binomial random variable.

**Definition 3.8** (The Beta-Binomial model). *The  $V$ -conditional probability of loss is*

$$\mathbb{P}[\ell_j = (a_j k + b_j)\delta \mid V = v] = \int_{[0,1]} \mathbb{P}[Z = k \mid p, n_j] \pi_j(p \mid V = v) dp$$

for any  $k = 0, \dots, n_j$ , where  $Z \sim \text{Bin}(n_j, p)$ , that is

$$\mathbb{P}[X = k \mid p, n_j] = \binom{n_j}{k} p^k (1-p)^{n_j-k}$$

and with the Beta distribution

$$\pi_j(p \mid V = v) = \frac{p^{\alpha(v)-1} (1-p)^{\beta(v)-1}}{\text{B}(\alpha(v), \beta(v))}$$

for some functions  $\alpha : [0, 1]^d \rightarrow \mathbb{R}_{+*}$  and  $\beta : [0, 1]^d \rightarrow \mathbb{R}_{+*}$ .

Conditional on  $V$  the number of loss units experienced upon default is the sum of a constant  $b_j$  and of  $k$  units  $a_j$  where  $k$  follows a Binomial distribution with parameter  $p$  and support  $0, \dots, n_j$ . In addition, the probability  $p$  is random and distributed according to a Beta distribution with parameters  $\alpha(v)$  and  $\beta(v)$ . Note that the functions  $\alpha$  and  $\beta$  may be obligor specific.

Although the Beta-binomial specification may look intimidating, it is a well-studied flexible distribution that nests a large spectrum of distributions such as the Bernoulli (see below), the

discrete uniform (when  $\alpha = \beta = 1$ ), and asymptotically the binomial (for large  $\alpha$  and  $\beta$ ). An additional important feature is that an explicit expression is available for its probability mass function

$$\mathbb{P}[\ell_j = (a_j k + b_j)\delta \mid V = v] = \frac{\Gamma(n_j + 1)}{\Gamma(k + 1)\Gamma(n_j - k + 1)} \frac{\Gamma(\alpha(v) + \beta(v))}{\Gamma(\alpha(v))\Gamma(\beta(v))} \\ \times \frac{\Gamma(k + \alpha(v))\Gamma(n_j - k + \beta(v))}{\Gamma(n + \alpha(v) + \beta(v))}.$$

for any  $k = 0, \dots, n_j$  and where  $\Gamma$  denotes the gamma function. The  $V$ -conditional loss amount mean and variance therefore also have a explicit expression

$$\mathbb{E}[\ell_j \mid V = v] = \left( a_j \frac{n_j \alpha(v)}{\alpha(v) + \beta(v)} + b_j \right) \delta$$

and

$$\text{Var}[\ell_j \mid V = v] = \frac{n_j \alpha(v) \beta(v) (\alpha(v) + \beta(v) + n_j)}{(\alpha(v) + \beta(v))^2 (\alpha(v) + \beta(v) + 1)} a_j^2 \delta^2.$$

Remark that the mean loss amount is positively correlated with  $V$  when the function  $v \mapsto \alpha(v)/(\alpha(v) + \beta(v))$  is increasing on  $[0, 1]$ .

**Example 3.9** (Bernoulli model). *The loss amount distribution reduces to a Bernoulli when  $n_j = 1$  with probability*

$$p(v) = \frac{\Gamma(\alpha(v) + \beta(v))}{\Gamma(\alpha(v))} \times \frac{\Gamma(1 + \alpha(v))}{\Gamma(1 + \alpha(v) + \beta(v))}$$

*which can take any value in  $(0, 1)$  and thus also be arbitrary close to the Dirac delta function.*

**Example 3.10** (Linear Beta-Binomial model). *Assume that  $d = 1$  and that the functions  $\alpha$ ,  $\beta$  are linear, then we have*

$$\alpha(v) = m_1 + m_2 v \\ \beta(v) = m_3 + m_4 v$$

*where  $m_i > 0$  for all  $i = 1, \dots, 4$ . This specification is discussed in further details in Section 4.4.*

## 4 Numerical analysis

In this section we illustrate the computational performance of our approach, and numerically study the properties of selected models with different dependence and loss given default assumptions.

### 4.1 Computational performance

We show here that the discrete Fourier transform (DFT) method proposed in Section 3 is significantly more efficient than the recursive methods suggested in Andersen et al. (2003); Hull and White (2004). Note that for a loss support of size  $M$  the DFT is computationally equivalent to the numerical inversion of Laurent and Gregory (2005) with  $M$  discretization points, yet the DFT returns the exact loss distribution.

We consider the standard one-factor and two-factor copula models. Figure 2 displays the computing time necessary to retrieve the probability mass function with the DFT and with the recursive method. The calculations have been performed on a single CPU from a standard personal computer in the R programming language. The DFT method is significantly faster than the recursive method in both cases: it takes roughly the same amount of time to retrieve a distribution with 1000 points with DFT and a 100 points with recursion.

## 4.2 Dependent defaults with a mixed copula

We investigate the joint default probability and the total number of defaults density in a one-factor copula model with a mixed bivariate copula specification as defined in Equation (6).

We fix  $K = 2$  and assume that  $p_{j,t} = 1 - e^{-\lambda t}$  for  $j \in \{1, 2\}$  with  $\lambda = 5\%$ . Consider the following copula mixture

$$C_{U_j, V}(u_j, v) = wC_{U_j, V}^C(u_j, v) + (1 - w)C_{U_j, V}^G(u_j, v)$$

for  $j \in \{1, \dots, N\}$ , for some  $w \in [0, 1]$ , and where  $C^C$  denotes the Clayton copula with parameter 5 and  $C^G$  the Gaussian copula with parameter 25%. Figure 3 displays the probability and cumulative density functions of joint defaults of two entities for the times  $0 \leq t \leq 20$ , and for the weights  $w \in \{0, 0.5, 1\}$ . The two limit cases therefore correspond to the Gaussian and Clayton copulas. We observe that the joint probability of default also becomes a mixture of the two limit cases.

We set  $N = 125$ , Figure 4 displays the total number of defaults at a 5-years horizon. It is visually obvious that the distribution of the number of defaults is a mixture of the two limit components: it has the bump of the Gaussian with parameter  $\rho = 25\%$  and the fat tail of the Clayton with parameter 5.

## 4.3 Credit derivatives

We explore the loss distribution of a large portfolio, a tranche on this portfolio, and a portfolio of tranches when the underlying tranches are independent and when they depend on the same factor  $V$ . Let  $N = 1000$  and assume that  $\ell_j = 1$  and  $\lambda_{jt} = 0.01$  for all  $j \in \mathcal{I}$  and  $t \geq 0$ . The reference model is the standard one-factor Gaussian copula with correlation parameter  $\rho = 25\%$ . All the tranches have for attachment point  $a_k = 100$  and detachment point  $b_k = 200$ . The CDO squared is composed of 10 tranches so as to have the same loss support as the portfolio.

Figure 5 displays the probability and cumulative mass functions of the portfolio, tranche, and portfolio of tranches at the 5-year horizon. Observe that the tranche loss distribution has two masses at the beginning and end of its support corresponding the probabilities of no loss and full loss respectively. These more concentrated masses combined and creates a spiky pattern in the portfolio of tranches loss distribution.

The CDO squared loss distribution has been computed under the assumption of unique factor and tranche specific factor. The two resulting loss distributions have dramatically different profiles. With independent factors the CDO squared appears even less exposed to losses than the vanilla portfolio. For example, the senior tranches on the pooled portfolio are virtually riskless. On the other hand, with a unique common factor the CDO squared has a fat tailed loss distribution and a large probability, about 91%, of having zero losses: when the risk driver behind all tranches is the same, the diversification benefit almost completely disappears. Similar results has been obtained Hull and White (2010) using Monte Carlo simulations.

#### 4.4 Stochastic and correlated loss amounts

In this section we investigate the impact of introducing stochastic losses that may be correlated with the factor  $V$  on the loss distribution of a portfolio. We consider the linear Beta-Binomial model presented in Section 3.3 and always assume that  $a_j = 1$  and  $b_j = 0$ .

Assume that  $m_3 = m_4 = 0$  such that the functions  $\alpha$  and  $\beta$  becomes constants numbers. Figure 6 displays the loss amount distribution for different parameters choices and shows that it presumably spans a large set of discrete distributions. We see that the attainable probability function may be flat, increasing, decreasing, or even bump shaped.

Assume now that  $m_3 = m_1$  and  $m_4 = m_2$  such that the  $V$ -conditional expected loss is

$$\mathbb{E}[\ell_j | V = v] = \frac{n_j a_j \delta (m_1 + m_2 (1 - v))}{2m_1 + m_2} + b_j \delta.$$

In this particular case the expected loss is constant is, that is

$$\mathbb{E}[\ell_j] = \int_0^1 \mathbb{E}[\ell_j | V = v] dv = \frac{1}{2}.$$

for any  $m_1$  and  $m_2$  when  $a_j = 1$  and  $b_j = 0$ . On Figure 7 we see that the  $V$ -conditional distribution of the loss amount may exhibit various different shapes, even-though the expected loss remains the same.

Consider the standard one-factor Gaussian copula with  $\rho = 0.25$ ,  $N = 125$ ,  $\lambda_j = 0.05$  for all  $j \in \mathcal{I}$ , and with the same loss amount model as above having an expected loss one half. Figure 8 shows that the loss distribution is significantly affected by the choice of dependence parameters. Compared to the benchmark case of independent and equi-distributed loss amounts, increasing the dependence on the factor  $V$  also increases the portfolio average loss and tail risk.

#### 4.5 Number of defaults and loss dependence

We investigate how dependent individual losses affect the portfolio loss distribution given a number of realized defaults. We remind that this distribution is required to price credit swaptions. Consider the usual one-factor homogeneous Gaussian copula with  $\rho = 0.25$ , with default intensities  $\lambda_{jt} = 0.05$ , with  $N = 125$  entities, and for a 5-year horizon. We assume that the  $V$ -conditional loss amounts  $\ell_j$  is given by

$$\ell_j = \begin{cases} 1 & \text{with probability } 1 - v \\ 0 & \text{with probability } v \end{cases}$$

such that  $\mathbb{E}[\ell_j] = 0.5$  for all  $j \in \mathcal{I}$ . The left panel on Figure 9 displays the probability density of the joint probability distribution of the number of defaults and loss  $(N_t, L_t)$  computed as described in Proposition 2.2. We observe that most of the probability mass is concentrated on a diagonal band near the origin, and that there is little to no mass on the off diagonal parts.

The right panel of Figure 9 displays the expected loss given a certain number of default, that is  $\mathbb{E}[L_t | N_t = n]$  for  $n = 0, \dots, 125$ . This value is increasing with  $N_t = n$  as the losses are expected to increase with the total number of defaults. Several interesting observations can be made. The marginal rate of losses starts from almost zero at the origin and increases rapidly, and the conditional expected loss converges to the maximal possible loss. This is in contrast with the case of independent loss amounts defined by  $\mathbb{P}[\ell_j = 0] = \mathbb{P}[\ell_j = 1] = 0.5$  also displays on this Figure and where the relation between  $N_t$  and  $L_t$  is linear.



## 5 Empirical analysis

In this section, we illustrate our approach by calibrating various factor copula models to credit index tranche prices.

### 5.1 Data

We focus on tranches of the *CDX.NA.IG* index, which is composed of 125 investment grade North American companies. Historically, all tranches except the most junior were unfunded. Similarly as standard swaps, they were quoted with a spread and didn't include upfront payments. Since 2009 however, a new set of rules, known as the *Big Bang Protocol*, was amended to the International Swaps and Derivatives Association's master agreement (i.e., the standardized contract used between dealers and their counterparties). Arguably the most important was the *100/500 Credit Derivative Initiatives*: by standardizing coupons at 1% or 5% per annum<sup>3</sup> with quarterly payments, the rule made the upfront necessary to enter a contract on any tranche.

Based on liquidity, new series of the *CDX.NA.IG* index with tenors of 3, 5, 7, and 10 years are determined every 6 months (in March and September). The series 21, issued in September 2013 with a tenor of 5 years, came along with four standardized tranches, whose spreads and attachments/detachments points are detailed in 1. Our sample contains 405 daily upfront payments for the four tranches, which we summarize in 2 and display in 10. By convention, the market quotes upfronts in percentage of the corresponding tranche width, which is about thirty times larger for the super-senior than for the equity. Furthermore, the sign of the upfront is also interesting: since it is negative, one most often receives money to buy protection on the super-senior tranche, as well as on the senior tranche at the end of the sample period.

### 5.2 Calibration

Let  $P^{a_i, b_i}$ ,  $a_i$  and  $b_i$  for  $i \in \{1, \dots, 4\}$  denote the quoted upfronts, and attachments/detachments points of each tranches. For a model parametrized with  $\theta \subseteq \Theta \subseteq \mathbb{R}^l$  (i.e.,  $l$  is the number of parameters), we denote by  $P^{a_i, b_i}(\theta)$  the model price, that is the quantity satisfying

$$P^{a_i, b_i}(\theta)(b_i - a_i) + V_{\text{prem}}^{a_i, b_i}(\theta) = V_{\text{prot}}^{a_i, b_i}(\theta),$$

where  $b_i - a_i$  is the tranche width, and the premium and protection legs are defined as

$$V_{\text{prem}}^{a_i, b_i}(\theta) = S^{a_i, b_i} \mathbb{E}_{\theta} \left[ \sum_{j=1}^n e^{-\int_0^{T_j} r_s ds} (T_j - T_{j-1}) \int_{T_{j-1}}^{T_j} \frac{b - a - \tau_t^{a, b}}{T_j - T_{j-1}} dt \right],$$

and

$$V_{\text{prot}}^{a_i, b_i}(\theta) = \mathbb{E}_{\theta} \left[ \int_0^T e^{-\int_0^t r_s ds} d\tau_t^{a_i, b_i} \right],$$

with  $0 = T_0 \leq \dots \leq T_n = T$  the payment dates,  $T$  the maturity, and  $S^{a_i, b_i}$  the tranche spread. See Appendix C for more details.

Assuming  $r = 0$  and a homogeneous portfolio with no recovery (i.e.,  $\delta_j = 0$ ), we let the default probability be

$$p_{j,t} = 1 - e^{-\lambda t}, \quad j \in \{1, \dots, 125\}$$

---

<sup>3</sup>Although it has since been extended to also include coupons of 0.25% and 10%.

where  $\lambda$  is the credit index swap spread, and the model is calibrated by minimizing the squared pricing error, that is

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \Theta}{\operatorname{argmin}} \sum_{i=1}^4 \left( P^{a_i, b_i} - P^{a_i, b_i}(\boldsymbol{\theta}) \right)^2. \quad (15)$$

In our current implementation, (15) is solved in two steps. First, we explore the parameter space to find a good starting value via a differential evolution algorithm. Second, we use the Nelder-Mead algorithm to refine the solution, enforcing the bounds by means of a parameter transformation.

### 5.3 Results

In Figure 11, we show calibration of various copulas to upfronts quoted on January 6th, 2014. While the one-factor Gaussian (dotted line) is completely off, both the one-factor  $t$  copula (dashed line) and the two-factors Gaussian-Clayton copula (grey line) perform better but miss the senior tranche. The only model achieving a perfect fit (i.e., the black line) is the following one-factor two-Gaussians mixture

$$C_{U_j, V}(u_j, v) = w C_{U_j, V}^{\rho_1}(u_j, v) + (1 - w) C_{U_j, V}^{\rho_2}(u_j, v), \quad j \in \{1, \dots, 125\}$$

with  $w \in [0, 1]$  and  $C^{\rho_i}$  is a Gaussian copula with parameter  $\rho_i$  for  $j \in \{1, 2\}$  (i.e.,  $\boldsymbol{\theta} = (w, \rho_1, \rho_2)$  and  $\Theta = [0, 1] \times [-1, 1] \times [-1, 1]$ ).

Repeating (15) of the mixture for each day of the sample, we obtain time-series of calibrated parameters that we display as the plain lines in Figure 12. There are two interesting observations that can be made. First, the parameters do not vary much over time, which indicates that the model is not over-parametrized and can be reliably estimated. Second, the second parameter is very close to 1, which means the second component of the mixture describes a comonotonic relationship between the factor and the uniform random variables for each obligor. In other words, we have

$$\mathbb{P}[U_j \leq u_j \mid V = v] \approx \begin{cases} w C_{U_j|V}^{\rho_1}(u_j \mid v), & \text{if } u_j \leq v, \\ w C_{U_j|V}^{\rho_1}(u_j \mid v) + (1 - w), & \text{otherwise} \end{cases}, \quad (16)$$

for  $j \in \{1, \dots, 125\}$ . When fixing  $\rho_2 = 0.99$  such that (16) holds, and calibrating  $\boldsymbol{\theta} = (w, \rho_1)$  only, similar results were obtained, and the parameters time series are the dotted lines in the upper-right panel of Figure 12.

In Figure 13, we display a model diagnostic for each of the four tranches. For each day in the sample period, the pricing errors, namely

$$P^{a_i, b_i} - P^{a_i, b_i}(\boldsymbol{\theta}_i) \text{ with } \begin{cases} \boldsymbol{\theta}_1 = (w, \rho_1, \rho_2) \\ \boldsymbol{\theta}_2 = (w, \rho_1, 0.99) \end{cases},$$

are the black and grey lines respectively for  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_2$ , and the bid-ask spread, that is

$$P_{ask}^{a_i, b_i} - P_{bid}^{a_i, b_i},$$

are the light grey lines. As the pricing errors are much lower than the bid-ask spread, the equity and mezzanine tranches are perfectly calibrated by both models. For the senior tranche with  $\boldsymbol{\theta}_2$  and the super-senior tranche however, the pricing errors and the bid-ask spread have the same order of magnitude. To alleviate this issue, we could switch the target of the minimization in the right-hand side of (15) from percentage of the tranche width to dollar amount. In other

words, by weighting each term of the sum by  $(b_i - a_i)^2$ , we would increase the relative importance of the super-senior tranche in the objective function. Nonetheless, the pricing error (and the bid-ask spread) are between 10 and 30 times smaller than the upfront itself.

To summarize, we achieve an almost perfect calibration to all tranches with only two parameters that remain stable over time.

## 6 Conclusion

We propose a new methodology paving the way for novel and promising applications in finance and insurance. Using factor copulas, we introduce a flexible and tractable class of reduced form models for dependent default times. Using bivariate copulas as building blocks, we extend our framework from one-factor to multi-factor specifications, and we show that our approach nests most standard models as special cases. Furthermore, assuming the distribution of individual losses given default to be discrete, we propose a method to compute explicitly and efficiently the distribution of the portfolio loss. This allows us to price complex multi-name credit derivatives such as credit index swaptions, tranches on a portfolio of loans, and tranches on a portfolio of tranches.

We illustrate the versatility and computational efficiency of our approach with numerical examples. In particular, we investigate the impact on the portfolio loss distribution of different default dependence assumptions. We also examine how the loss distributions of credit derivatives, such as tranche and CDO squared, are affected. We calibrate multiple models to credit index tranche prices. We show that a particular specification achieve almost perfect calibration to all tranches using only two parameters that are stable over time.

This work focused on the static construction of dependent default times, taking as given the deterministic marginal default probabilities. Ongoing research therefore aims at constructing a dynamic framework that remains tractable. With a doubly stochastic construction of default times, this may be achieved by choosing stochastic default intensities and factor copulas so that the characteristic function of the portfolio loss remains computable. Another more applied research direction is concerned with the exploration of new default dependence structures, in particular for multi-factor copula models.

## A Proofs

### Proof of Lemma 2.2

The joint probability of default rewrites

$$\begin{aligned}\mathbb{P}[\tau_1 \leq t_1, \dots, \tau_N \leq t_N] &= \mathbb{P}[U_1 \leq p_{1,t_1}, \dots, U_N \leq p_{N,t_N}] \\ &= C_U(p_{1,t_1}, \dots, p_{N,t_N})\end{aligned}$$

where the second line follows by definition of  $C_U$ .

### Proof of Proposition 2.3

Observe that for all  $j = 1, \dots, N$  the random vector  $(U_j, V)$  takes values on  $[0, 1]^2$  and has uniform marginal densities, this implies that

$$\mathbb{P}[U_j \leq u_j, V \leq v] = C_{U_j, V}(u_j, v)$$

for some bivariate copulas  $C_{U_j, V}$  and any  $(u_j, v) \in [0, 1]^2$ . Therefore we have

$$\mathbb{P}[U_j \leq u_j \mid V = v] = C_{U_j|V}(u_j \mid v)$$

and by plugging this into Equation (5) then integrating with respect to the density  $f_V(v) = v$  of  $V$  we obtain

$$\begin{aligned}C_U(u_1, \dots, u_N) &= \int_0^1 \prod_{j=1}^N \mathbb{P}[U_j \leq u_j, V \leq v] f_V(v) dv \\ &= \int_0^1 \prod_{j=1}^N C_{U_j, V}(u_j, v) dv.\end{aligned}$$

The desired expression then follows from Lemma 2.2.

### Proof of Proposition 2.6

The joint default probability conditional on the default of the  $k \in \mathcal{D}$  entities and  $V$  rewrites

$$\begin{aligned}&\mathbb{P}[\tau_1 \leq t_1, \dots, \tau_N \leq t_N \mid \{\tau_k = t_k : k \in \mathcal{D}\} \cup \{V = v\}] \\ &= \mathbb{P}[U_1 \leq p_{1,t_1}, \dots, U_N \leq p_{N,t_N} \mid \{U_k = p_{k,t_k} : k \in \mathcal{D}\} \cup \{V = v\}] \\ &= \prod_{k \in \mathcal{D}} \frac{\partial}{\partial u_k} \prod_{j=1}^N \frac{\partial}{\partial v} C_{U_j, V}(p_{j,t_j}, v) \\ &= \prod_{j \in \mathcal{D}} \frac{\partial^2}{\partial u_j \partial v} C_{U_j, V}(p_{j,t_j}, v) \prod_{j \in \mathcal{I}} \frac{\partial}{\partial v} C_{U_j, V}(p_{j,t_j}, v) \\ &= \prod_{j \in \mathcal{D}} c_{U_j, V}(p_{j,t_j}, v) \prod_{j \in \mathcal{I}} C_{U_j|V}(p_{j,t_j} \mid v).\end{aligned}$$

Equation (7) follows by integrating with respect to the density  $f_V(v) = v$  of  $V$  on its support  $[0, 1]$ .

## Proof of Propostion 2.7

The  $V$ -conditional joint default probability as a similar expression as in Equation (4). The unconditional joint default probability follows by integrating with respect to the joint density  $c_V(v)$  of  $V$  which gives the expression for  $C_U$  as  $dC_V(v) = c_V(v)dv$ . Observe now that the joint distribution of the random vector  $(U_j, V)$  is by construction given by a  $(1+d)$ -dimensional copula  $C_{U_j, V}$  for all  $j \in \mathcal{I}$ . By definition we must have

$$\begin{aligned} C_{U_j, V}(u_j, v) &= \mathbb{P}[U_j \leq u_j, V \leq v] \\ &= \int_0^{v_1} \dots \int_0^{v_d} \mathbb{P}[U_j \leq u_j \mid V = y] d\mathbb{P}[V \leq y] \\ &= \int_0^{v_1} \dots \int_0^{v_d} C_{U_j|V}(u_j \mid y) dC_V(y) \end{aligned}$$

for all  $(u_j, v) \in [0, 1]^{1+d}$  which gives Equation (8).

## Proof of Corollary 2.8

The density of  $V$  is given by  $C_V(v) = \prod_{j=1}^d v_j$ , and following Joe (1996) the conditional copulas are given by

$$C_{U_j|V}(u_j \mid v) = \frac{\partial C_{U_j, V_k|V_{-k}}(C_{U_j|V_{-k}}(u_j \mid v_{-k}), v_k \mid v_{-k})}{\partial v_k}$$

for any  $k = 1, \dots, d$ , and where  $V_{-k} = (V_1, \dots, V_{k-1}, V_{k+1}, \dots, V_d)$  denotes the random vector  $V$  without its  $k$ -th coordinate. By iterating the previous equation, the conditional copula  $C_{U_j|V}(u_j \mid v)$  can be rewritten as a recursive composition of bivariate linking copulas

$$C_U(u_1, \dots, u_n) = \int_{[0,1]^d} \prod_{j=1}^N C_{U_j|V_1}(\cdot \mid v_1) \circ \dots \circ C_{U_j|V_d}(u_j \mid v_d) dv$$

where  $C_{U_j, V_k}$  denotes a bivariate copula for  $j = 1, \dots, N$  and  $k = 1, \dots, d$ .

## Proof of Theorem 2.9

Observe that the random vector  $U = (F_{Y_1}(Y_1), \dots, F_{Y_N}(Y_N))$  and  $V = (F_{X_1}(X_1), \dots, F_{X_d}(X_d))$  have uniform margins by construction suggesting that their distributions are given by copulas. The following theorem proves the existence of  $C_V$ .

**Theorem A.1** (Sklar's Theorem 1959).  *$F_V$  is a joint distribution with margins  $F_{X_i}$  for  $i \in \{1, \dots, d\}$  if and only if there exists a copula  $C_V$ , that is a distribution which is supported in the unit hypercube and has uniform margins, such that*

$$F_X(x_1, \dots, x_N) = C_V(F_{X_1}(x_1), \dots, F_{X_N}(x_N))$$

for all  $x \in \mathbb{R}^N$ . Moreover, if the margins are continuous, then  $C_V$  is unique.

For all  $v \in [0, 1]^d$  the theorem implies that

$$\begin{aligned} C_V(v_1, \dots, v_d) &= F_X(F_{X_1}^{-1}(v_1), \dots, F_{X_d}^{-1}(v_d)) \\ &= \mathbb{P}[X_1 \leq F_{X_1}^{-1}(v_1), \dots, X_d \leq F_{X_d}^{-1}(v_d)] \\ &= \mathbb{P}[F_{X_1}(X_1) \leq v_1, \dots, F_{X_d}(X_d) \leq v_d] \\ &= \mathbb{P}[V_1 \leq v_1, \dots, V_d \leq v_d]. \end{aligned}$$

The copula  $C_V$  is thus the joint distribution of probability integral transforms. The  $X$ -conditional independence of  $Y$  implies that

$$\mathbb{P}[Y_1 \leq y_{1t_1}, \dots, Y_N \leq y_{Nt_N} \mid X = x] = \prod_{j=1}^N F_{Y_j|X}(y_{jt_j} \mid x),$$

where  $F_{Y_j|X}$  denotes the distribution of  $Y_j$  conditional on  $X$  such that

$$\mathbb{P}[\tau_j \leq t_j \mid X = x] = \mathbb{P}[U_j \leq p_{j,t_j} \mid V = v],$$

where  $v = \tilde{F}_X(x) := (F_{X_1}(x_1), \dots, F_{X_d}(x_d))$ . A copula representation of the above probability can finally be obtained by applying the conditional equivalent of Sklar's theorem:

**Theorem A.2** (Patton's Theorem 2002).  *$F_{Y|X}$  is a joint conditional distribution with conditional margins  $F_{Y_i|X}$  for  $i \in \{1, \dots, N\}$  if and only if there exists a conditional copula  $C_{U|V}$ , that is a conditional distribution which is supported in the unit hypercube and has uniform conditional margins, such that*

$$F_{Y|X}(y_1, \dots, y_N \mid x) = C_{U|V}(F_{Y_1|X}(y_1 \mid x), \dots, F_{Y_N|X}(y_N \mid x) \mid F_X(x))$$

for all  $y \in \mathbb{R}^N$  and  $x \in \mathbb{R}$ . Moreover, if the conditional margins are continuous, then  $C_{U|V}$  is unique.

For all  $u \in [0, 1]^N$  and  $v \in [0, 1]^d$  the theorem implies

$$\begin{aligned} C_{U|V}(u_1, \dots, u_N \mid v) &= F_{Y|X}\left(F_{Y_1|X}^{-1}(u_1), \dots, F_{Y_N|X}^{-1}(u_N) \mid \tilde{F}_X^{-1}(v)\right) \\ &= \mathbb{P}\left[Y_1 \leq F_{Y_1|X}^{-1}(u_1), \dots, Y_N \leq F_{Y_N|X}^{-1}(u_N) \mid X = \tilde{F}_X^{-1}(v)\right] \\ &= \mathbb{P}\left[F_{Y_1|X}(Y_1 \mid X) \leq u_1, \dots, F_{Y_N|X}(Y_N \mid X) \leq u_N \mid \tilde{F}_X(X) = v\right] \\ &= \mathbb{P}[U_1 \leq u_1, \dots, U_N \leq u_N \mid V = v]. \end{aligned}$$

In other words, the copula  $C_{U|V}$  is also the joint conditional distribution of the conditional probability integral transforms. As such, the joint conditional distribution of default times is given by

$$\begin{aligned} \mathbb{P}[\tau_1 \leq t_1, \dots, \tau_N \leq t_N \mid X = F_X^{-1}(v)] &= \mathbb{P}[U_1 \leq p_{1,t_1}, \dots, U_N \leq p_{N,t_N} \mid V = v] \\ &= C_{U|V}(p_{1,t_1}, \dots, p_{N,t_N} \mid v), \end{aligned}$$

which completes the proof.

### Proof of Proposition 3.1

The default times and the loss amounts being independent conditional on  $V$  we have

$$\begin{aligned} \mathbb{E}\left[e^{iuL_t\delta^{-1}} \mid V = v\right] &= \mathbb{E}\left[e^{iu\sum_{j=1}^N \mathbb{1}\{\tau_j \leq t\} \ell_j \delta^{-1}} \mid V = v\right] \\ &= \prod_{j=1}^N \mathbb{E}\left[e^{iu\mathbb{1}\{\tau_j \leq t\} \ell_j \delta^{-1}} \mid V = v\right] \end{aligned}$$

Furthermore, by independence of the random variables  $\mathbb{1}_{\{\tau_j \leq t\}}$  and  $\ell_j$  conditional on  $V$  we have

$$\begin{aligned} \mathbb{E} \left[ e^{iu \mathbb{1}_{\{\tau_j \leq t\}} \ell_j \delta^{-1}} \mid V = v \right] &= 1 - \mathbb{P}[\tau_j \leq t \mid V = v] \\ &\quad + \mathbb{P}[\tau_j \leq t \mid V = v] \phi_{\ell_j}(u, v) \end{aligned}$$

where  $\phi_{\ell_j}(u, v) := \mathbb{E} \left[ e^{iu \ell_j \delta^{-1}} \mid V = v \right]$  denotes the  $V$ -conditional characteristic function of  $\ell_j \delta^{-1}$ . We finally apply the tower property

$$\begin{aligned} \phi_{L_t}(u) &= \mathbb{E} \left[ \mathbb{E} \left[ e^{iu L_t \delta^{-1}} \mid V = v \right] \right] \\ &= \int_{[0,1]^d} \mathbb{E} \left[ e^{iu L_t \delta^{-1}} \mid V = v \right] dC_V(v) \\ &= \int_{[0,1]^d} (1 - p_{j,t}(v) + p_{j,t}(v) \phi_{\ell_j}(u, v)) dC_V(v) \end{aligned}$$

where  $C_V$  is the density of  $X$ , and  $p_{j,t}(v) = C_{U_j|V}(p_{j,t} \mid v)$ .

### Proof of Lemma 3.2

The proof is a straightforward application of discrete Fourier transform inversion. Observe that the random variable  $L_t \delta^{-1}$  has state space  $\{0, 1, \dots, M\}$ . Its discrete Fourier transform is given by

$$F_m = \sum_{k=0}^M \mathbb{P}[L_t \delta^{-1} = k] e^{-i \frac{2\pi m k}{M+1}} = \phi_{L_t} \left( \frac{-2\pi m}{(M+1)} \right)$$

where  $\phi_{L_t}$  as in Proposition 3.1 is the characteristic function of  $L_t \delta^{-1}$ . The probability mass function can be recovered as follows

$$\mathbb{P}[L_t = k\delta] = \frac{1}{M+1} \sum_{m=0}^M F_m e^{i \frac{2\pi m k}{M+1}}.$$

Equation (11) follows by observing that the signs can equivalently be switched between the complex weights.

### Proof of Proposition 3.4

The proof of this proposition is immediate from the factor copula construction with proportional losses.

### Proof of Corollary 3.5

This follows directly from Proposition 3.1 and Proposition 3.4.

## Proof of Proposition 3.6

By construction we have

$$\begin{aligned}
F_{x,y} &:= \phi_{N_t, L_t}(\mu x, \nu y) \\
&= \mathbb{E} \left[ \prod_{j=1}^N \exp \left\{ i \mathbb{1}_{\{\tau_j \leq t\}} (\mu x + \nu y \ell_j \delta^{-1}) \right\} \right] \\
&= \mathbb{E} \left[ \exp \left\{ \sum_{j=1}^N i \mathbb{1}_{\{\tau_j \leq t\}} (\mu x + \nu y \ell_j \delta^{-1}) \right\} \right] \\
&= \mathbb{E} \left[ e^{i\mu x N_t + i\nu y L_t \delta^{-1}} \right]
\end{aligned}$$

Using this last expectation and the explicit expressions for  $\mu$  and  $\nu$  we obtain

$$F_{x,y} = \sum_{j=0}^N \sum_{k=0}^M \mathbb{P}[N_t = j, L_t = \delta k] e^{i \frac{2\pi j}{N+1} x} e^{i \frac{2\pi k}{M+1} y}.$$

This last expression is the two dimensional discrete Fourier transform of the density of the variable  $(N_t, L_t \delta^{-1})$ . The density can then immediately be retrieved by applying the inverse two-dimensional discrete Fourier transform inversion as follows

$$\mathbb{P}[N_t = j, L_t = \delta k] = \sum_{x=0}^N \sum_{y=0}^M F_{x,y} e^{-i \frac{2\pi x}{N+1} j} e^{-i \frac{2\pi y}{M+1} k}.$$

## B Standard copula models

We derive in this appendix the factor copula representation of the most popular models that have been proposed in the literature on multi-name credit risk.

### Gaussian copula models

Let us denote the Gaussian copula and  $h$ -function by

$$C_{U,V}^G(u, v; \rho) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho)$$

and

$$C_{U,V}^G(u | v; \rho) = \Phi \left( \frac{\Phi^{-1}(u) - \rho \Phi^{-1}(v)}{1 - \rho^2} \right),$$

where  $\Phi(\cdot)$  is the standard normal distribution and  $\Phi_2(\cdot, \cdot; \rho)$  is the bivariate normal distribution with correlation  $\rho$ . For instance, when  $d = 1$  and all bivariate copulas are Gaussian, then a representation for the joint distribution of default times is the copula of a 1-factor model

$$Y_j = \beta_j X + \sqrt{1 - \beta_j^2} Z_j,$$

where  $X, Z_1, \dots, Z_N$  are i.i.d.  $N(0, 1)$  random variables. In this case, the correlation parameter for the bivariate copula linking the default of obligor  $j$  to the systematic factor is  $\beta_j$ . By considering



a unique correlation parameter  $\beta_j = \rho$  for  $j \in \{1, \dots, N\}$ , Li (2000) is a special case of our formulation. Furthermore, when  $d > 1$ , then a representation for the joint distribution of default times is the copula of a  $d$ -factor model

$$Y_j = \sum_{i=1}^p \beta_{j,i} X_i + Z_j,$$

where  $X_1, \dots, X_p, Z_1, \dots, Z_N$  are i.i.d.  $N(0, 1)$  random variables. In this case, the parameters for the second to  $d$  factors are partial correlations, namely

$$\begin{aligned} \rho_{U_j, V_k | X_1, \dots, V_{k-1}} &= \frac{\text{Cov}(Y_j, X_k | X_1, \dots, X_{k-1})}{\sqrt{\text{Var}(Y_j | X_1, \dots, X_{k-1})} \sqrt{\text{Var}(X_k | X_1, \dots, X_{k-1})}} \\ &= \frac{\beta_{j,k}}{\sqrt{1 - \beta_{j,1}^2 - \dots - \beta_{j,k-1}^2}}. \end{aligned}$$

### Stochastic correlation models

It is straightforward to build more complex factor models, stochastic correlations models are obtained by writing

$$Y_j = (B_j \alpha_j + (1 - B_j) \beta_j) X + \sqrt{1 - (B_j \alpha_j + (1 - B_j) \beta_j)^2} Z_j,$$

where  $B_j$  are i.i.d. Bernoulli( $b_j$ ) and  $X, Z_1, \dots, Z_N$  as before. For this model, the bivariate copulas are convex sum of Gaussian copulas, that is

$$\begin{aligned} C_{U_j, V}^{SC}(u_j, v; \alpha_j, \beta_j, b_j) &= b_j^2 C_{U_j, V}^G(u_j, v; \alpha_j^2) \\ &\quad + 2b_j(1 - b_j) C_{U_j, V}^G(u_j, v; \alpha_j \beta_j) \\ &\quad + (1 - b_j)^2 C_{U_j, V}^G(u_j, v; \beta_j^2), \end{aligned}$$

and deriving the  $h$ -function yields

$$C_{U|V}^{SC}(u | v; \alpha_j, \beta_j, b_j) = b_j C_{U|V}^G(u | v; \alpha_j) + (1 - b_j) C_{U|V}^G(u | v; \beta_j).$$

### The $t$ -Student model

Usually,  $t$ -student models are specified by considering,

$$Y_j = \sqrt{W} \left( \beta_j X + \sqrt{1 - \beta_j^2} Z_j \right)$$

where  $W$  is an i.i.d. random variable such than  $\nu/W$  is  $\chi^2(\nu)$  and  $X, Z_1, \dots, Z_N$  as before. Then the default times are independent conditional on  $(W, X)$  and their conditional probability distribution is easily derived (see e.g. Burtschell et al. (2009)). Using our formulation, we obtain an equivalent  $t$ -student model by considering the copula and  $h$ -function directly, that is

$$C_{U,V}^t(u, v; \rho, \nu) = t_2(t_\nu^{-1}(u), t_\nu^{-1}(v); \rho, \nu)$$

and

$$C_{U,V}^t(u, v; \rho, \nu) = t_{\nu+1}(f(u, v)), \text{ with } f(u, v) = \frac{t_\nu^{-1}(u) - \rho t_\nu^{-1}(v)}{\sqrt{\frac{(1-\rho^2)(\nu + (t_\nu^{-1}(v))^2)}{\nu+1}}},$$

where  $t_\nu(\cdot)$  is the  $t$ -student distribution with  $\nu$  degrees of freedom and  $t_2(\cdot, \cdot; \rho, \nu)$  is the bivariate  $t$ -student distribution with correlation  $\rho$  and degrees of freedom  $\nu$ . Compared to the other formulation, our alternative only require a one-dimensional integration. Furthermore, using different degrees of freedom for each bivariate copulas offers additional modeling flexibility without additional cost.

## The Archimedean models

One-parameter archimedean copulas are built by considering a continuous, strictly decreasing and convex generator  $\psi : [0, 1] \times \Theta \rightarrow [0, \infty)$  such that  $\psi(1; \theta) = 0$  for all  $\theta \in \Theta$ , where  $\Theta$  represents the parameter space. Using this generator, a bivariate copula is obtained by writing

$$C_{U,V}^\psi(u, v; \theta) = \psi^{-1}(\psi(u; \theta) + \psi(v; \theta); \theta).$$

For such a copula, the  $h$ -function  $C_{U,V}^\psi$  is usually straightforward to derive, and we summarize the most popular in Tables 3 and 4.

## The Gaussian mixture

The Gaussian mixture model from Li and Liang (2005) is a one-factor copula mixture as described in Equation 6. The bivariate copulas are furthermore assumed to be Gaussian and equal  $C_{U_j,V}^k = C_{U_l,V}^k$  for any  $j, l = 1, \dots, N$  and  $k = 1, \dots, K$ .

## C Pricing Formulas

The credit contracts described in this appendix are composed of two cash-flows series. The contract buyer pays predefined coupons to the seller at the payments dates  $0 = T_0 \leq \dots \leq T_n = T$  where  $T$  is the contract maturity, we call this series of cash-flow the premium leg  $V_{\text{prem}}$ . The contract seller pays default contingent cash-flows to the buyer at the defaults dates when losses materialize, we call this series of cash-flow the protection leg  $V_{\text{prot}}$ . The contract value for the buyer is then given by  $V_{\text{prot}} - V_{\text{prem}}$ . We denote  $r_t$  the time- $t$  risk-free rate and derive the contracts values at the initial date.

### Tranche

A credit swap on a tranche  $\mathcal{T}^{a,b}$  with attachment point  $a$  and detachment point  $b$  is a protection insuring the loss experienced on the tranche, in exchange of scheduled premium payments proportional to the remaining size of the tranche. The value of the protection leg is

$$V_{\text{prot}} = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r_s ds} d\mathcal{T}_t^{a,b} \right]$$

where the tranche density  $\mathcal{T}_t^{a,b}$  is defined in Equation (12). The value of the premium leg is

$$V_{\text{prem}} = S^{a,b} \mathbb{E} \left[ \sum_{j=1}^n e^{-\int_0^{T_j} r_s ds} (T_j - T_{j-1}) \int_{T_{j-1}}^{T_j} \frac{b - a - \mathcal{T}_t^{a,b}}{T_j - T_{j-1}} dt \right]$$

where  $S^{a,b}$  is the tranche spread. In practice, the above expressions for the two legs are necessarily approximated by replacing the integrals with sums where quadrature and trapezoid methods cab

be used. Assuming that the short-rate and the default times are uncorrelated, Mortensen (2006) used the parsimonious following discretization

$$V_{\text{prot}} \approx \sum_{j=1}^n B\left(\frac{t_j + t_{j-1}}{2}\right) \left(\mathbb{E}\left[\mathcal{T}_{t_j}^{a,b} - \mathcal{T}_{t_{j-1}}^{a,b}\right]\right)$$

and

$$V_{\text{prem}} \approx S^{a,b} \sum_{j=1}^n (t_j - t_{j-1}) B(t_j) \left(b - a - \frac{\mathbb{E}\left[\mathcal{T}_{t_j}^{a,b} + \mathcal{T}_{t_{j-1}}^{a,b}\right]}{2}\right)$$

where  $B(t)$  denotes the risk-free bond price with maturity  $t$  and notional equal to one.

### Credit swap

A credit swap on an index pays the realized losses in exchange for scheduled premium payments proportional to the number of non-defaulted entities. The value of the protection leg is

$$V_{\text{prot}} = \mathbb{E}\left[\sum_{j=1}^N e^{-\int_0^{\tau_j} r_s ds} \ell_j\right] = \mathbb{E}\left[\int_0^T e^{-\int_0^t r_s ds} dL_t\right]$$

where the total loss  $L_t$  is defined in Equation (10). The value of the premium leg is

$$V_{\text{prem}} = S \mathbb{E}\left[\sum_{j=1}^n e^{-\int_0^{T_j} r_s ds} (T_j - T_{j-1})(N - N_t) + \int_{T_{j-1}}^{T_j} e^{-\int_0^u r_s ds} (u - T_{j-1}) dN_u\right]$$

where  $S$  is the index spread and  $N_t$  is the time- $t$  total number of default defined in Equation (14).

### Credit swaption

A European credit swaption offers the right to enter a credit swap at a future time  $0 < T_m < T$  at a predefined index spread  $S^*$ . In addition, the credit swaption typically provides default protection between the issuance date 0 and the option maturity  $T_m$ . Denote  $V_{\text{CS}}(T_m)$  the time- $T_m$  value of the credit swap which follows directly from above. The time- $T_m$  value of the credit swaption is given by

$$V_{\text{CSO}} = \mathbb{E}\left[e^{-\int_0^{T_m} r_s ds} \left(V_{\text{CS}}(T_m) + (L_{T_m} - L_0)\right)^+\right].$$

Name	Attachement	Detachment	Spread
Equity	0	3	5
Mezzanine	3	7	1
Senior	7	15	1
Super-senior	15	100	0.25

Table 1: Tranches structure on the CDX.NA.IG.21.

The attachment/detachment points and the spread per annum for each of the four tranches are given in percentage.

	Equity	Mezzanine	Senior	Super-senior
Mean	15.18	5.92	-0.29	-0.23
Vol	4.10	2.74	1.27	0.22
Min	8.59	1.34	-2.07	-0.53
Max	24.87	13.28	2.82	0.25

Table 2: Summary statistics for the tranches on the CDX.NA.IG.21.

The statistics concern the upfront payments, which are quoted in percentage of the tranche width.

	Generator $\psi$	Inverse generator $\psi^{-1}$	Parameter space $\Theta$
Clayton	$\frac{u^{-\theta}-1}{\theta}$	$(1+\theta u)^{-1/\theta}$	$(0, \infty)$
Gumbel	$(-\log(u))^\theta$	$\exp(-u^{1/\theta})$	$[1, \infty)$
Frank	$-\log\left(\frac{\exp(-\theta u)-1}{\exp(-\theta)-1}\right)$	$-\frac{1}{\theta} \log(1 + \exp(-t)(\exp(-\theta)-1))$	$(-\infty, \infty) \setminus \{0\}$
Joe	$-\log(1 - (1-u)^\theta)$	$1 - (1 - \exp(-u))^{1/\theta}$	$[1, \infty)$
Independence	$-\log(u)$	$\exp(-u)$	$\emptyset$

Table 3: **Archimedean copulas:** the generator  $\psi$ , the inverse generator  $\psi^{-1}$  and the parameter space  $\Theta$ .

	Copula $C_{U,V}^\psi$
Clayton	$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$
Gumbel	$e^{-((-\log(u))^\theta + (-\log(v))^\theta)^{1/\theta}}$
Frank	$-\frac{1}{\theta} \log \left( \frac{1-e^{-\theta} - (1-e^{-u\theta})(1-e^{-v\theta})}{1-e^{-\theta}} \right)$
Joe	$1 - ((1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta)^{1/\theta}$
Independence	$uv$
	$h$ -function $C_{U V}^\psi$
Clayton	$C_{U,V}^\psi(u, v; \theta) v^{-1-\theta}$
Gumbel	$C_{U,V}^\psi(u, v; \theta) \frac{((-\log(u))^\theta + (-\log(v))^\theta)^{1/\theta-1} (-\log(v))^\theta}{v \log(v)}$
Frank	$\frac{e^\theta (e^{\theta u} - 1)}{e^{\theta u + \theta v} - e^{\theta u + \theta} - e^{\theta v + \theta} + e^{\theta v}}$
Joe	$(C_{U,V}^\psi(u, v; \theta))^{1-\theta} (1-v)^{\theta-1} (1-(1-u)^\theta)$
Independence	$u$

Table 4: **Archimedean copulas:** the copula  $C_{U,V}^\psi$  and the  $h$ -function  $C_{U|V}^\psi$ .

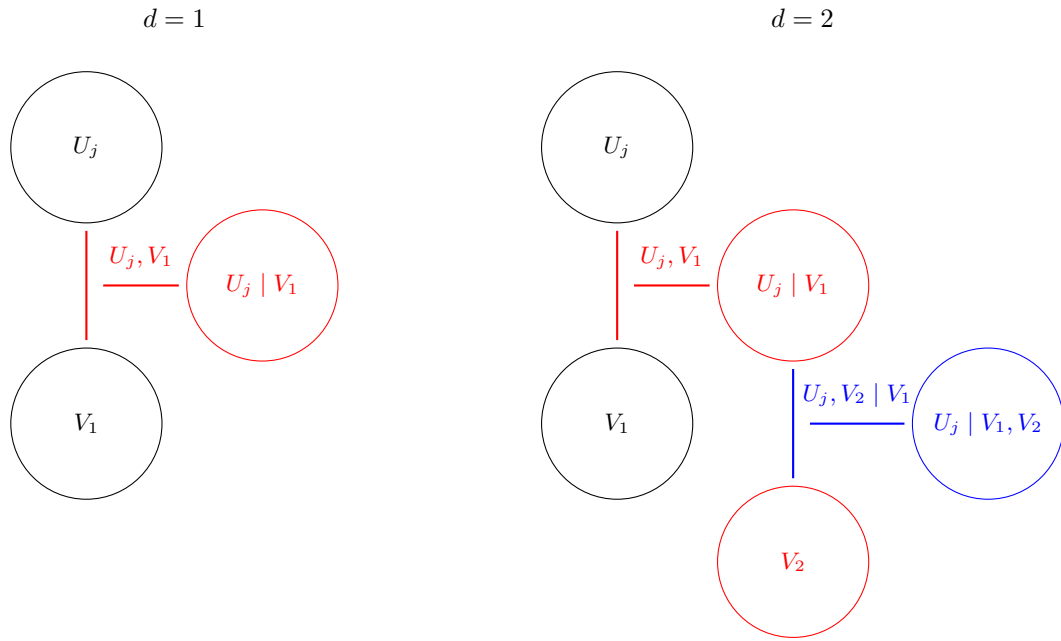


Figure 1: Pair-copula dependence decomposition.

The dependence structure is constructed from left to right: circles surround variables and lines connect variables whose dependence is specified with a bivariate copula.



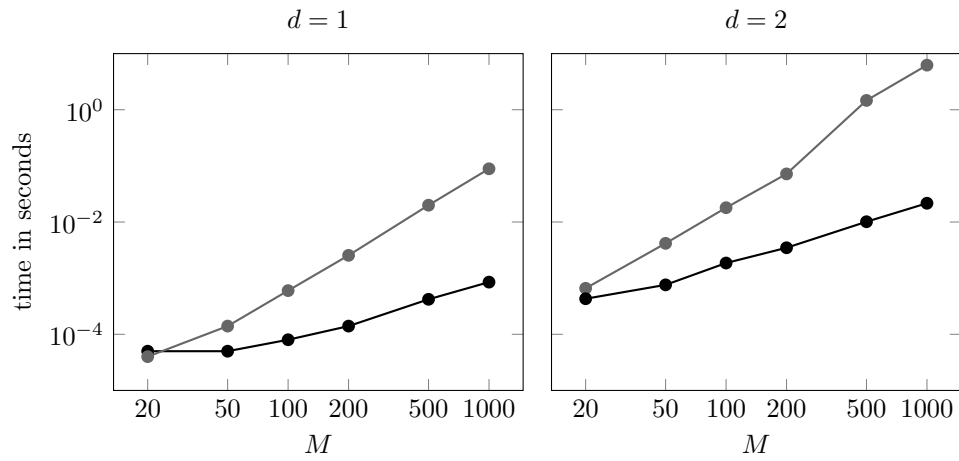


Figure 2: Computation performance.

The time in seconds to compute the loss probability mass function is displayed against the loss support size  $M$  for the discrete Fourier transform (black line) and recursive (grey line) methods. The one-factor (left panel) and two-factor (right panel) standard Gaussian copula have been used under the assumption of constant loss given default.

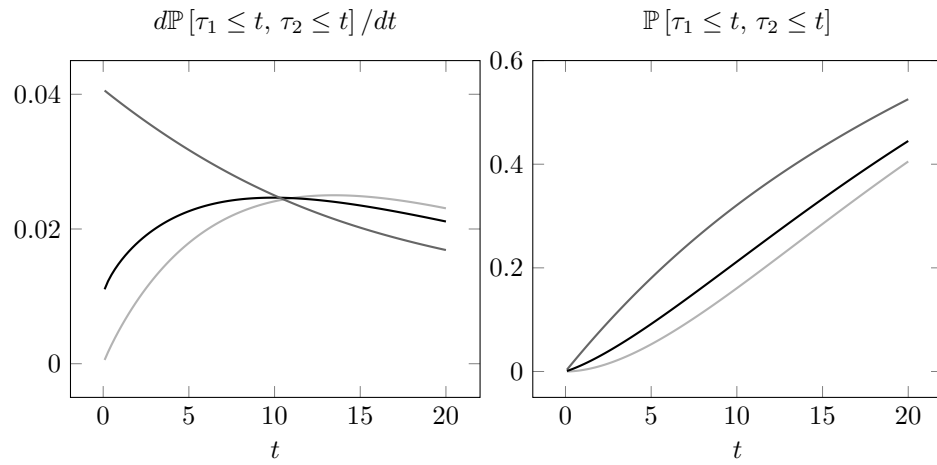


Figure 3: Defaults dependence and copula mixture.

The probability (left panel) and cumulative (right panel) density functions of the joint default are displayed for time horizons ranging from 1 week to 20 years for three different one-factor models: an equiweighted copula mixture (black line) between a Gaussian copula with  $\rho = 0.25$  (light-grey line) and a Clayton copula with parameter equal to 5 (grey line).

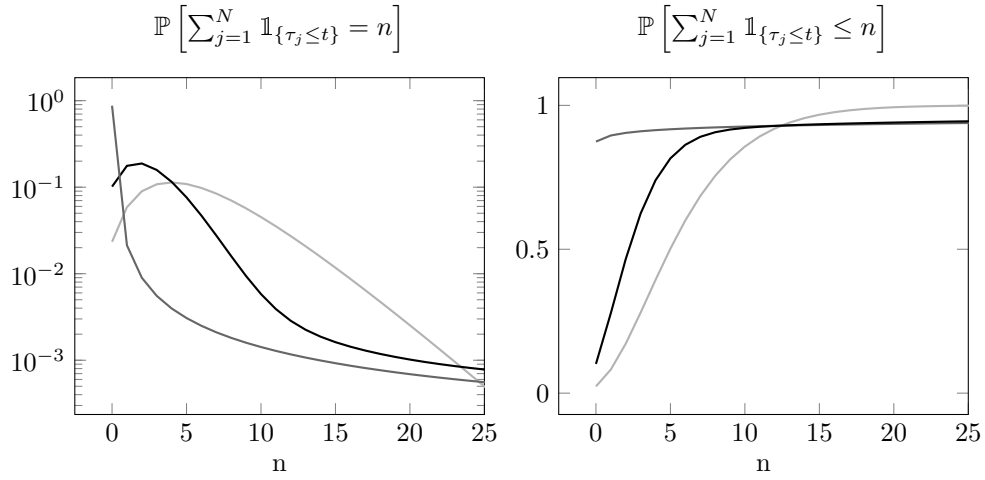


Figure 4: Total number of defaults with copula mixture.

The Figure displays the probability (left panel) and cumulative (right panel) density functions of the total number of defaults on a portfolio of 125 homogeneous entities. We assume that  $p_{j,t} = 5\%$  for all  $j \in \mathcal{I}$  and consider an equiweighted copula mixture (black line) between a Gaussian copula with  $\rho = 0.25$  (light-grey line) and a Clayton copula with parameter equal to 5 (grey line).

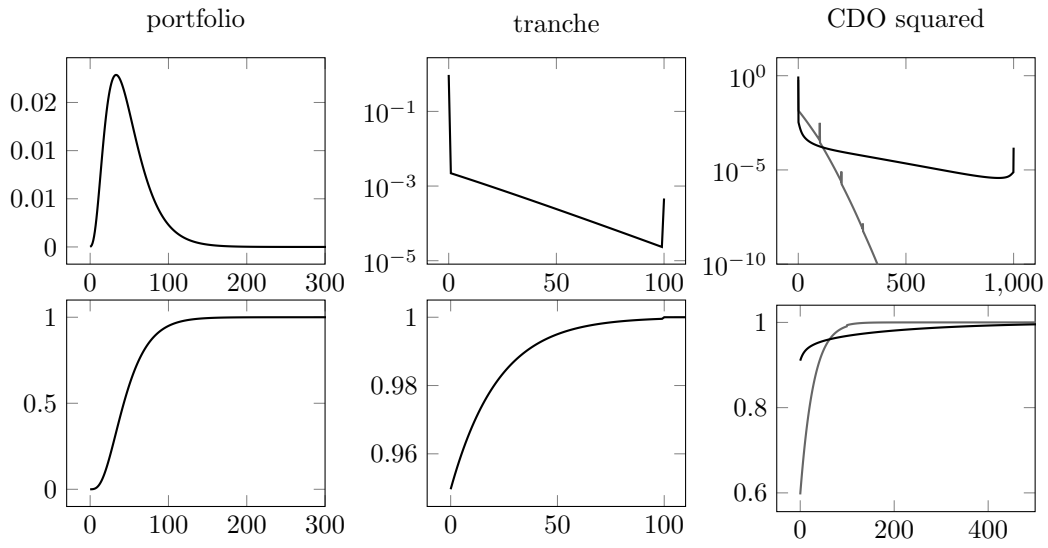


Figure 5: Multi-name credit derivatives losses.

The probability (first row) and cumulative (second row) density functions of the loss distribution are displayed for three different derivatives. The first column is concerned with a portfolio of 1000 entities, the second column with a tranche on this portfolio with attachment point 100 and detachment point 200, and the third column with a portfolio of 10 such tranches coming from different portfolios with a unique risk factor (black line) and with independent risk factors (grey line).

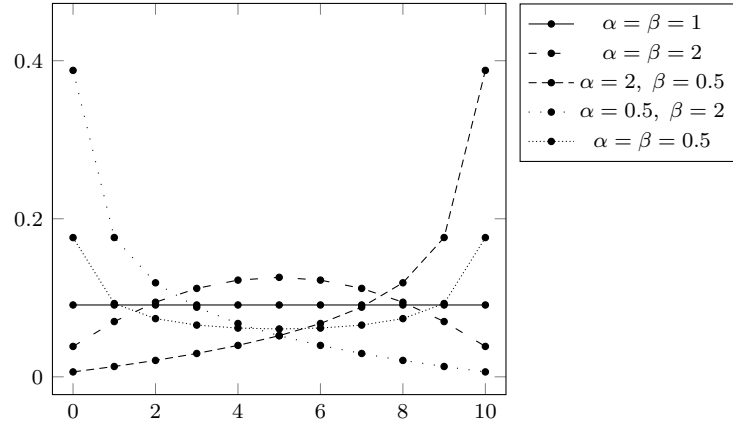


Figure 6: Beta-Binomial distribution.

The distribution of the Beta-Binomial random variables with  $n = 10$  is displayed for different parameters choices.

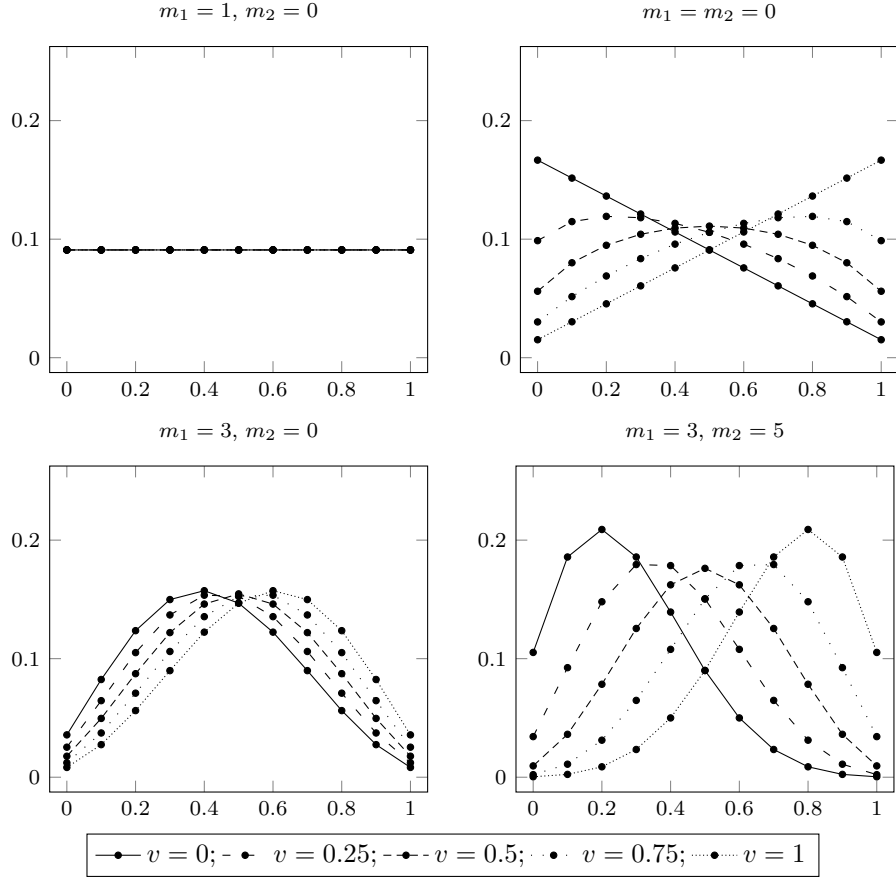


Figure 7: Beta-Binomial distribution.

The distribution of the Beta-Binomial random variables with  $n = 10$  is displayed for different parameters choices. The Beta-Binomial loss distribution for a single obligor:  $m_1 = 1$  and  $m_2 = 0$  (top left),  $m_1 = 1$  and  $m_2 = 1$  (top right),  $m_1 = 3$  and  $m_2 = 1$  (bottom left),  $m_1 = 3$  and  $m_2 = 5$  (bottom right). In each panel, the distribution is represented for different values of the systematic factor.

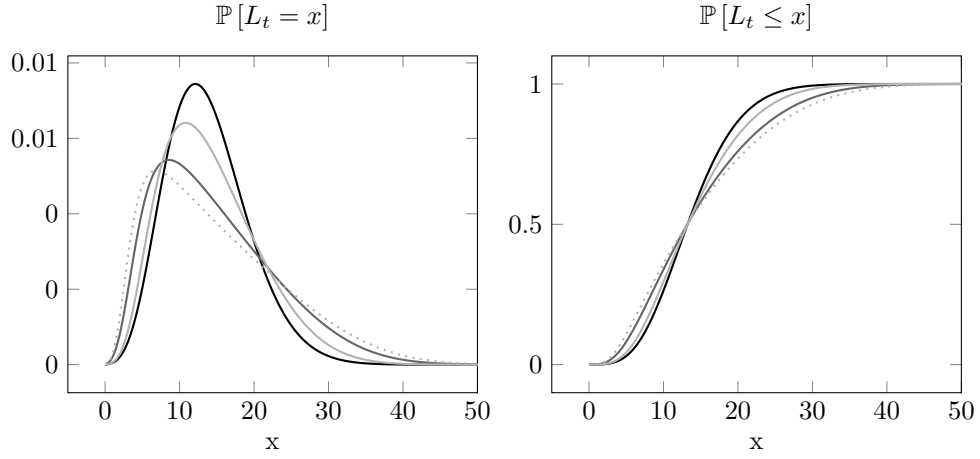


Figure 8: Loss distribution and loss amounts dependence.

The probability (first row) and cumulative (second row) density functions of the loss distribution are displayed for three different loss amounts specifications. With a standard one-factor copula model with  $\rho = 0.25$ , constant marginal default intensity  $\lambda_{jt} = 5\%$ , and a 5-year horizon we consider the linear Beta-Binomial loss amounts with  $m_1 = m_3$  and  $m_2 = m_4$  for the values:  $m_1 = 1$  and  $m_2 = 0$  (black line),  $m_1 = 1$  and  $m_2 = 1$  (grey line),  $m_1 = 3$  and  $m_2 = 1$  (light-grey line), and  $m_1 = 3$  and  $m_2 = 5$  (dotted light-grey line).

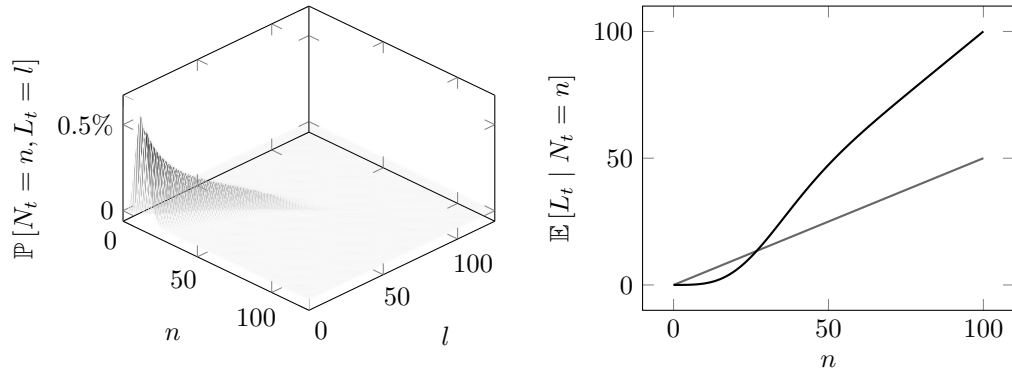


Figure 9: Number of defaults and loss dependence.

The left panel displays the joint probability density of the number of default  $N_t$  and the loss  $L_t$ . The right panel displays the expected loss given  $n$  defaults, the loss amounts may not depend from the factor  $V$  (grey line) or may depend on the factor  $V$  (black line). The reference model is a one-factor homogeneous Gaussian copula with  $\rho = 0.25$ , default intensities  $\lambda_{jt} = 0.05$ , a 5-year horizon, and contains  $N = 125$  entities. The loss amount  $\ell_{jt}$  is zero or one and has an expected value of 50%.



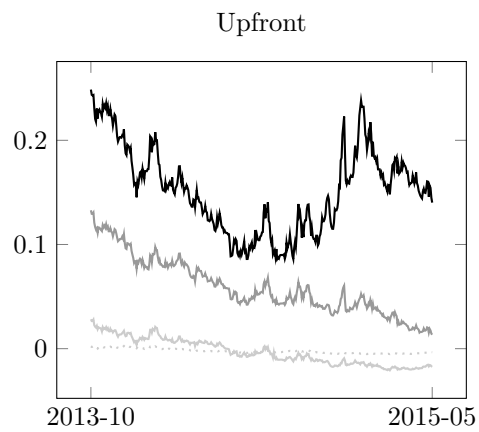


Figure 10: Upfronts on CDX.NA.IG.21 tranches.

The time-series of quoted upfront payments are displayed for the equity (black line), mezzanine (grey line), senior (light-grey line) and super-senior (dotted light-grey line) tranches.

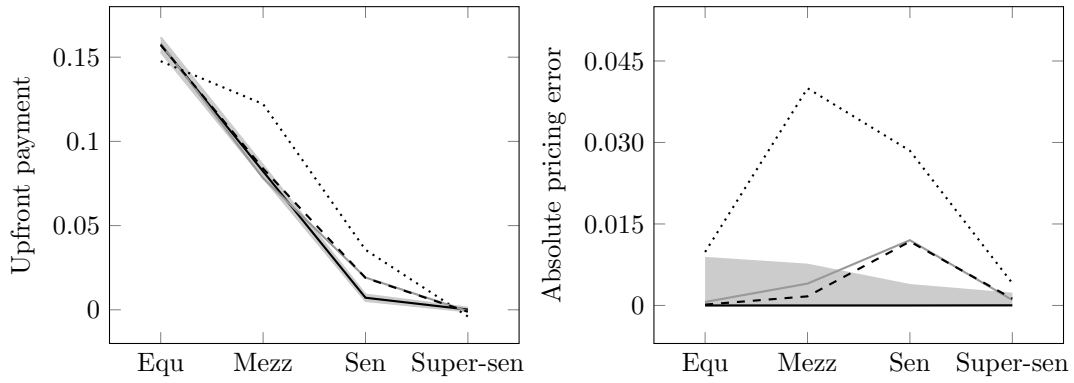


Figure 11: Models calibration to tranches on the CDX.NA.IG.21.

Using the quoted upfronts on January 6th, 2014, various copulas models are calibrated: the one-factor Gaussian (dotted line), the one-factor  $t$  copula (dashed line), the two-factors Gaussian-Clayton copula (grey line), and the one-factor mixture with two Gaussians (black line). The shaded area is the bid-ask spread.

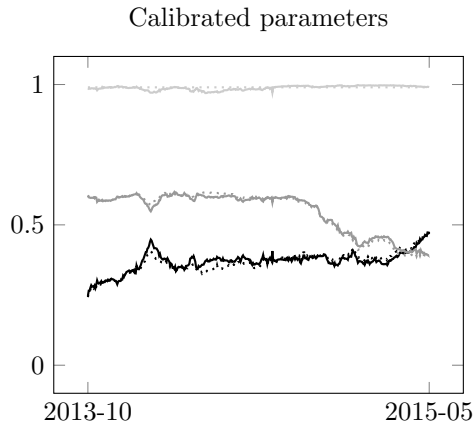


Figure 12: Parameters calibrated on CDX.NA.IG.21 tranches. The time-series of calibrated parameters are displayed for  $w$  (black lines)  $\rho_1$  (grey lines) and  $\rho_2$  (light-grey lines). The plain and dotted lines correspond to models with either three (plain) or two (dotted) parameters, that with either  $\theta_1 = (w, \rho_1, \rho_2)$  or  $\theta_2 = (w, \rho_1, 0.99)$ .

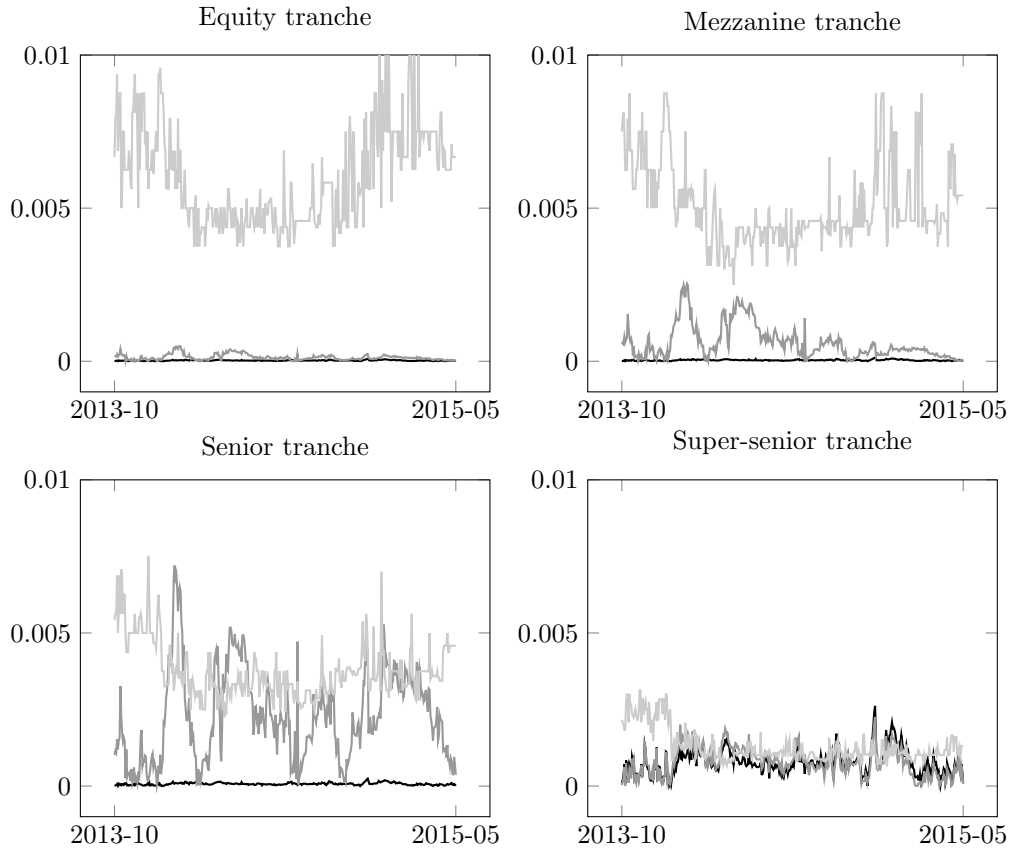


Figure 13: Diagnostic of models calibrated on CDX.NA.IG.21 tranches.

Model diagnostic are displayed with the bid-ask spread (light-grey line) and the pricing errors for the model with either three (black line) or two (grey line) parameters, that with either  $\theta_1 = (w, \rho_1, \rho_2)$  or  $\theta_2 = (w, \rho_1, 0.99)$ .

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